Duality and anti-duality for allocation rules in economic problems I: an axiomatic analysis

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March 31, 2018

*The idea presented in this paper was developed during my visit to the University of Rochester in 2014. I am grateful to William Thomson for his helpful suggestions and discussions. I also thank Shin Sakaue for his comments. I am financially supported by Grant-in-Aid for Scientific research (C), Project #17K03629. Of course, any remaining errors are mine.

Abstract

We develop the notions of duality and anti-duality for axiomatic analysis of allocation rules. First, we show basic properties of duality and anti-duality for allocation rules. Next, using the notion of duality and axioms involved in axiomatizations of the Shapley rule for airport problems, we axiomatize the Shapley rule for bidding ring problems. Finally, using the notion of anti-duality and axioms involved in axiomatizations of the nucleolus for airport problems, we axiomatize the nucleolus rule for bidding ring problems. From the approach proposed, we may derive appropriate interpretations of axioms involved in axiomatizations of economic rules.

Keywords: duality; anti-duality; axiomatization; Shapley rules; nucleolus rules

JEL classification: C69, C71

1 Introduction

"Claims problems" are well-known allocation problems in economics. They deal with the situation where the liquidation value of a bankrupt firm has to be allocated between its creditors, but there is not enough to honor the claims of all creditors. The problem is to determine how the creditors should share the liquidation value (O'Neill 1982; Thomson 2003, for a survey of the literature). In claims problems, Thomson and Yeh (2008) introduce operators on the space of division rules and uncover the underlying structure of the space of division rules. The notion of "duality" plays an important role in their analysis. For each claims problem, this notion gives us its dual problem. Intuitively speaking, the dual of a claims problem is to determine how the creditors should abandon some part of their claims. Also, the notion of duality is applied to division rules: Given a division rule for claims problems, its dual rule is the same division rule for their dual problems. A division rule is said to be "self-dual", if the outcome chosen by this division rule and the outcome chosen by its dual rule always coincide with each other.

Analogously to claims problems, one can define "dual solutions" and "self-dual solutions" in cooperative game theory. Intuitively speaking, given a solution for coalitional games with transferable utility (TU games, for short), its dual solution is defined as the same solution for the dual games.[1] A solution is said to be "self-dual", if the outcome chosen by this solution and the outcome chosen by its dual solution always coincide with each other. These concepts can help us to uncover a hidden structure of solutions, axioms, and

[1] The notion of "dual games" is well known in the literature on cooperative game theory. The definition of dual games and their interpretation are stated in Section 2.

axiomatizations on the domain of all TU games (Funaki 1998). Analogously to the notions of dual solutions and self-dual solutions, one can also define the notions of “anti-dual solutions” and “self-anti-dual solutions” in cooperative game theory (Oishi and Nakayama 2009; Oishi et al. 2016).[2] Oishi et al. (2015) uncover another hidden structure of solutions, axioms, and axiomatizations on several classes of TU games. However, the potential of the notions of duality and anti-duality to economic analysis has not been developed.

The purpose of this paper is to propose an analytical framework for axiomatizations of allocation rules for economic problems. Toward this purpose, we develop the unified approach of duality and anti-duality for axiomatic analysis of allocation rules. This new approach enables us to derive axiomatizations of allocation rules for some economic problems, which have not been analyzed in the existing literature, and to uncover a hidden relationship between distinct rules.

The framework proposed has the following advantages: Axiomatizations of allocation rules depend on economic problems under investigation. So, axiomatizations of an allocation rule for economic problems cannot be derived from those of the same rule for other problems. On the other hand, we offer a general linkage, “duality” and “anti-duality” relations, between axioms involved in axiomatizations of allocation rules for economic problems. Thanks to the duality and anti-duality relations, we can derive axiomatizations of an allocation rule for some economic problems from those of the same rule for other problems. Thus, an axiomatization of allocation rules for some problems, which has not been analyzed in the existing literature, is possible automatically.

First, we show basic properties of (anti-)dual axioms and of (anti-)dual axiomatizations of allocation rules. That is, we verify that an axiom for the (anti-)dual of a rule can be derived from taking the (anti-)dual of an axiom for the original rule. We also verify that the (anti-)dual of a rule can be axiomatized by taking the (anti-)dual of the axioms involved in an axiomatization of the original rule.

Next, we apply the notion of duality to “airport problems”, and “bidding ring problems”. One can apply the notion of anti-duality to these problems, and then one can obtain the same result derived from the duality approach. For simplicity of our analysis, we take the duality approach. Airport problems are cost sharing problems of an airstrip among airlines (Littlechild and Owen 1973; Thomson 2007, for a survey of the literature). A bidding ring problem (Graham et al. 1990) describes the situation where bidders form a ring in a single-object English auction. The ring reduces or eliminates buyer competition, thereby securing an advantage over the seller. The problem forced by the members of the ring is to share the benefit of their strategy.

[2] The definition of anti-dual games, anti-dual solutions and self-anti-dual solutions is stated in Section 2.

The "Shapley rule" is a mapping on some domain of allocation problems that associates with each problem in the domain the "Shapley value" of the corresponding TU game. The Shapley value (Shapley 1953) is the most important single-valued solution of TU games with economic applications. For instance, Chun et al. (2012) axiomatize the Shapley rule for airport problems. Considering these axioms in a duality relation, it is shown that one can easily axiomatize the Shapley rule for bidding ring problems. These axioms are self-duals to those appearing in Chun et al. (2012), and an interpretation of the axioms is possible in the almost same manner as in Chun et al. (2012).

Finally, we apply the notion of anti-duality to analysis of the "nucleolus rules" for bidding ring problems. The nucleolus rule is a mapping on some domain of allocation problems that associates with each problem in the domain the "nucleolus" of the corresponding TU game. The nucleolus (Schmeidler 1969) is another important single-valued solution of TU games with economic applications. For instance, Hwang and Yeh (2012) axiomatize the nucleolus for airport problems. Considering these axioms in an anti-duality relation, it is shown that one can easily axiomatize the nucleolus for bidding ring problems. These axioms are self-anti-duals to those appearing in Hwang and Yeh (2012). However, a consistency property referred to as "bidding-ring consistency" axiom can be interpreted in a different way since it is obtained as a quietly different form from "airport consistency" appearing in Hwang and Yeh (2012). Thus, from the approach proposed, we may derive appropriate interpretations of axioms involved in axiomatizations of economic rules.

The rest of this paper is organized as follows. In Section 2, we explain the notions of duality and anti-duality for cooperative game theory. In Section 3, we introduce the notions of duality and anti-duality for allocation rules, and show basic properties of these notions. In Section 4, using duality, we axiomatize the Shapley rule for bidding ring problems. In Section 5, using anti-duality,we axiomatize the nucleolus rule for bidding ring problems.

2 Preliminaries

There is a universe of potential agents, denoted $\mathcal{I} \subseteq \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.[3] Let $\mathcal{N}$ be the class of non-empty and finite subsets of $\mathcal{I}$, and $N \in \mathcal{N}$. A **coalitional game with transferable utility for $\boldsymbol{N}$** (a **TU game for $\boldsymbol{N}$**, for short) is a function $v\colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. A set $S \in 2^N$ is called a **coalition**. As far as there is no confusion, we sometimes denote by i instead of $\{i\}$ a singleton. For all $S \in 2^N$, $v(S)$ represents what coalition S can achieve on its own. Let $\boldsymbol{\mathcal{V}^N}$ be the **class of TU games for $\boldsymbol{N}$**, and $\mathcal{V} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{V}^N$.

[3]We use $\subseteq$ for weak set inclusion, and $\subset$ for strict set inclusion.

Let $\mathbb{R}^N$ denote the Cartesian product of $|N|$ copies of $\mathbb{R}$, indexed by the members of N. A **payoff vector** for N is an element x of $\mathbb{R}^N$. For all $x \in \mathbb{R}^N$ and all $S \in 2^N$, let $x_S = (x_i)_{i \in S}$.

A **solution**, denoted $\boldsymbol{\varphi}$, is a mapping defined on some domain of games that associates with each game in the domain a non-empty set of payoff vectors. A solution is **single-valued** if it associates with each game in its domain a unique payoff vector.

Given a game v for N, the **dual of $\boldsymbol{v}$**, denoted $\boldsymbol{v^d}$, is defined by setting, for all $S \subseteq N$,

$$v^d(S) \equiv v(N) - v(N \backslash S).$$

The number $v^d(S)$ is the amount that the complementary coalition $N \backslash S$ cannot prevent S from obtaining.

Let $\mathcal{V}$ be a class of games such that if $v \in \mathcal{V}$, then $v^d \in \mathcal{V}$. Given a solution φ on $\mathcal{V}$, the **dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^d}$, is defined by setting, for all $v \in \mathcal{V}$,

$$\varphi^d(v) \equiv \varphi(v^d).$$

A solution φ on $\mathcal{V}$ is **self-dual** if for all $v \in \mathcal{V}$, $\varphi(v) = \varphi^d(v)$.

An **axiom** of a solution is a property that should be satisfied by the solution. **Two axioms are dual to each other** if whenever a solution satisfies one of them, the dual of this solution satisfies the other. That is, two axioms A and A' are dual to each other if for all solutions that satisfy A, it holds that their duals satisfy A', and conversely, for all solutions that satisfy A', it holds that their duals satisfy A. **An axiom is self-dual** if it is its own dual.

Given a game v for N, the **anti-dual of $\boldsymbol{v}$**, denoted $\boldsymbol{v^{ad}}$, is defined by setting, for all $S \subseteq N$,

$$v^{ad}(S) \equiv -v^d(S).$$

Let $\mathcal{V}$ be a class of games such that if $v \in \mathcal{V}$, then $v^{ad} \in \mathcal{V}$. Given a solution φ on $\mathcal{V}$, the **anti-dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^{ad}}$, is defined by setting, for all $v \in \mathcal{V}$,

$$\varphi^{ad}(v) \equiv -\varphi(v^{ad}).$$

A solution φ on $\mathcal{V}$ is **self-anti-dual** if for all $v \in \mathcal{V}$, $\varphi(v) = \varphi^{ad}(v)$. **Two axioms are anti-dual to each other** if whenever a solution satisfies one of them, the anti-dual of this solution satisfies the other. That is, two axioms A and A' are anti-dual to each other if for all solutions that satisfy A, it holds that their anti-duals satisfy A', and conversely, for all solutions that satisfy A', it holds that their anti-duals satisfy A. **An axiom is self-anti-dual** if it is its own anti-dual.

In Figure 1, the horizontal arrows show the opposite-sign relation. For instance, v^{ad} is v^d with the opposite sign. In the left picture, the vertical arrow shows the duality relation. For instance, v^d is *dual* of v, and v is *dual* of

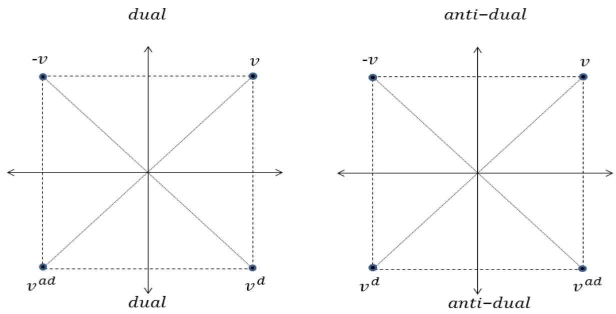

Figure 1: **Left**: duality relation between dual and anti-dual games. **Right**: anti-duality relation between dual and anti-dual games.

v^d. Similarly, v^{ad} is *dual* of $-v$, and $-v$ is *dual* of v^{ad}. In the right picture, the vertical arrow shows the anti-duality relation. For instance, v^{ad} is *anti-dual* of v, and v is *anti-dual* of v^{ad}. Similarly, v^d is *anti-dual* of $-v$, and $-v$ is *anti-dual* of v^d.

Finally, we introduce well-known *single-valued* solutions for TU games. The **Shapley value** (Shapley 1953) is defined as follows: for all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, and all $i \in N$,

$$Sh_i(v) \equiv \sum_{\substack{S \subseteq N \\ S \not\ni i}} \frac{|S|!(|N|-|S|-1)!}{|N|!} \big[v(S \cup \{i\}) - v(S)\big].$$

Given $N \in \mathcal{N}$ and $v \in \mathcal{V}^N$, let $I(v)$ be the set of vectors $x \in \mathbb{R}^N$ such that for all $i \in N$, $x_i \geq v(\{i\})$, and $\sum_N x_i = v(N)$. Let $\mathcal{V}^N$ be a class of games such that for all $v \in \mathcal{V}^N$, $I(v) \neq \emptyset$. For all $x \in I(v)$, let $e(v,x) \in \mathbb{R}^{2^N}$ be defined by setting, for all $S \subseteq N$, $e_S(v,x) \equiv v(S) - \sum_S x_i$. For all $z \in \mathbb{R}^{2^N}$, $\theta(z) \in \mathbb{R}^{2^N}$ is defined by rearranging the coordinates of z in non-increasing order. For all $z \in \mathbb{R}^{2^N}$, ***z* is lexicographically smaller than *z′*** if $\theta_1(z) < \theta_1(z')$ or $[\theta_1(z) = \theta_1(z')$ and $\theta_2(z) < \theta_2(z')]$ or $[\theta_1(z) = \theta_1(z')$ and $\theta_2(z) = \theta_2(z')$ and $\theta_3(z) < \theta_3(z')]$, and so on. The **nucleolus** (Schmeidler 1969) is defined as follows:

$$Nu(v) \equiv \left\{ x \in I(v) \;\middle|\; \begin{array}{c} \text{For all } y \in I(v)\backslash\{x\},\ e(v,x) \text{ is} \\ \text{lexicographically smaller than } e(v,y) \end{array} \right\}.$$

The nucleolus is a *single-valued* solution.

3 Duality and anti-duality for allocation rules, and their basic properties

In this section, we introduce the notions of duality and anti-duality for allocation rules. We also show properties of (anti-)dual axioms and of (anti-)dual axiomatizations of allocation rules.

3.1 Duality and anti-duality for allocation rules

An **allocation problem for $\boldsymbol{N}$** is a pair (N, p), where $N \in \mathcal{N}$ is a finite non-empty set of agents and $p = (p_i)_{i \in N}$ is a profile of parameters for N. For each $i \in N$, the parameter p_i is the benefit or the cost experienced by agent $i \in N$ when engaging in some economic activity. Let $\mathcal{P}$ be the set of all allocation problems on $\mathcal{N}$.

Given all $S \in 2^N$, we denote by $v_P : \mathcal{P} \rightarrow \mathbb{R}^{2^N}$ a mapping that associates with each allocation problem (N, p) in the domain the unique $2^{|N|}$-dimensional vector whose S-component is the amount coalition S can obtain on its own. By convention, $v_P(N, p)(\emptyset) = 0$. The number $v_P(N, p)$ is the **coalitional game for $\boldsymbol{N}$ derived from the allocation problem $\boldsymbol{(N, p)}$**.

Let $\boldsymbol{\mathcal{V}_P}$ be the class of all coalitional games derived from allocation problems $\mathcal{P}$. Given $(N, p) \in \mathcal{P}$, an **allocation** for (N, p) is a vector $x \in \mathbb{R}^N$ such that $\sum_N x_i = v_P(N, p)(N)$. Let $\mathbf{X}\,\boldsymbol{(N, p)}$ be the set of allocations for (N, p). A **solution for coalitional games** is a mapping $\phi : \mathcal{V}_P \rightarrow \mathbb{R}^N$ that associates with each coalitional game $v_P(N, p)$ in the domain a unique allocation in $X(N, p)$. We refer to the composite mapping $\varphi \equiv \phi \circ v_P$ as an **allocation rule, or simply a rule, for allocation problems on the domain of $\boldsymbol{\mathcal{P}}$**. For instance, we refer to the composite mapping $\varphi \equiv Sh \circ v_P$ as the **Shapley rule**, and to the composite mapping $\varphi \equiv Nu \circ v_P$ as the **nucleolus rule**.

Given a rule φ on $\mathcal{P}$, **the dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^d}$, is defined by setting, for all $(N, p) \in \mathcal{P}$,

$$\varphi^d(N, p) \equiv \phi[(v_P^d)(N, p)].$$

A rule φ on $\mathcal{P}$ is **self-dual** if for all $(N, p) \in \mathcal{P}$, $\varphi(N, p) = \varphi^d(N, p)$. **Two axioms are dual to each other** if whenever a rule satisfies one of them, the dual of this rule satisfies the other. That is, two axioms A and A' are dual to each other if for all rules that satisfy A, it holds that their duals satisfy A', and conversely, for all rules that satisfy A', it holds that their duals satisfy A. **An axiom is self-dual** if it is its own dual.

Given a rule φ on $\mathcal{P}$, **the anti-dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^{ad}}$, is defined by setting, for all $(N, p) \in \mathcal{P}$,

$$\varphi^{ad}(N, p) \equiv -\phi[(v_P^{ad})(N, p)].$$

A rule φ on $\mathcal{P}$ is **self-anti-dual** if for all $(N,p) \in \mathcal{P}$, $\varphi(N,p) = \varphi^{ad}(N,p)$. **Two axioms are anti-dual** if whenever a rule satisfies one of them, the anti-dual of this rule satisfies the other. That is, two axioms A and A' are anti-dual to each other if for all rules that satisfy A, it holds that their anti-duals satisfy A', and conversely, for all rules that satisfy A', it holds that their anti-duals satisfy A. **An axiom is self-anti-dual** if it is its own anti-dual.

3.2 Basic properties of (anti-)dual axioms for rules

3.2.1 Propositional functions, axioms, and axiomatizations

We show basic properties of duality and anti-duality for rules. In order to show these properties, we introduce the mathematical structure which explicitly deals with (anti-)dual axioms for rules for allocation problems. The basic idea of this mathematical structure steams from Funaki (1998). The mathematical structure proposed by Funaki (1998) deals with dual axioms for solutions for TU games, not (anti-)dual axioms for rules for allocation problems.

First, we introduce the mathematical structure for dual axioms for allocation rules using the notations appearing in the subsection 3.1.

Given a class $\mathcal{P}$, a class $\mathcal{V}_P$ and a solution ϕ on $\mathcal{V}_P$, a **propositional function *F*** is generically defined as follows:

$$F : \{((N,p), \varphi(N,p)) : (N,p) \in \mathcal{P},\ \varphi(N,p) \in X(N,p)\} \to \{0,1\},$$

where $\varphi(N,p) \equiv \phi \circ v_P(N,p)$ for all $(N,p) \in \mathcal{P}$ and all $v_P \in \mathcal{V}_P$. We say that the propositional function F with respect to (N,p) and φ is **true** (resp. **false**) if $F((N,p), \varphi(N,p)) = 1$ (resp. $F(\cdot,\cdot) = 0$).

Let $\mathcal{F}$ be the set of all propositional functions. Given a class $\mathcal{P}$, a class $\mathcal{V}_P$ and a solution ϕ on $\mathcal{V}_P$, an **equivalence relation** $\sim_\varphi$ on $\mathcal{F}$ is defined as follows:

$$F \sim_\varphi \bar{F} \quad \Leftrightarrow \quad F((N,p), \varphi(N,p)) = \bar{F}((N,p), \varphi(N,p)) \ \text{ for all } (N,p) \in \mathcal{P}.$$

Given a propositional function $\bar{F} \in \mathcal{F}$ on $\mathcal{P}$, an **axiom for a rule φ with respect to $\bar{F}$** is defined by setting, for all $(N,p) \in \mathcal{P}$,

$$E_{\bar{F}}(\mathcal{P}, \varphi) \equiv \{F : F \sim_\varphi \bar{F}\}.$$

A **rule φ satisfies the axiom $E_{\bar{F}}(\mathcal{P}, \varphi)$ with respect to $\bar{F}$** if for all $(N,p) \in \mathcal{P}$, and all $F \in E_{\bar{F}}(\mathcal{P}, \varphi)$, $F((N,p), \varphi(N,p)) = 1$.

On $\mathcal{P}$, a **rule φ is axiomatized by the set of axioms** if the rule φ satisfies a set of axioms with respect to some propositional functions and any other solutions do not satisfy it.

3.2.2 Dual axioms, and dual axiomatizations

Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and a propositional function $F \in \mathcal{F}$, the **dual of** F, denoted F^d, is defined by setting, for all $(N,p) \in \mathcal{P}$,

$$F^d((N,p),\varphi(N,p)) \equiv F((N,p),\varphi^d(N,p)),$$

where $\varphi^d(N,p) = \phi \circ v_P^d(N,p)$ for all $(N,p) \in \mathcal{P}$ and all $v_P \in \mathcal{V}_P$.

Given a propositional function $\bar{F} \in \mathcal{F}$, a **dual axiom for a rule φ with respect to $\bar{F}$**, denoted $E^d_{\bar{F}}(\mathcal{P},\varphi)$, is defined by setting, for all $(N,p) \in \mathcal{P}$,

$$E^d_{\bar{F}}(\mathcal{P},\varphi) \equiv E_{\bar{F}^d}(\mathcal{P},\varphi).$$

The following theorem shows that one can derive an axiom for the dual of a rule from taking the dual of an axiom for the original rule.

Theorem 1 *(Existence Theorem of dual axioms for rules) Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and a propositional function $F \in \mathcal{F}$, a rule φ satisfies an axiom $E_F(\mathcal{P},\varphi)$ if and only if the dual rule φ^d satisfies the dual axiom $E^d_F(\mathcal{P},\varphi^d)$.*

Proof. Let φ^d be a rule satisfying an axiom $E^d_F(\mathcal{P},\varphi^d)$, that is, for all $(N,p) \in \mathcal{P}$ and all $G \in E^d_F(\mathcal{P},\varphi^d)$,

$$G((N,p),\varphi^d(N,p)) = 1.$$

By the duality of propositional functions,

$$\begin{aligned}
& E^d_F(\mathcal{P},\varphi^d) \\
= \;& \{G : G((N,p),\varphi^d(N,p)) = F^d((N,p),\varphi^d(N,p)) \text{ for all } (N,p) \in \mathcal{P}\} \\
= \;& \{G : G((N,p),\phi \circ v^d(N,p)) = F^d((N,p),\phi \circ v^d(N,p)) \text{ for all } (N,p) \in \mathcal{P}\} \\
= \;& \{G : G^d((N,p),\varphi(N,p)) = F((N,p),\varphi(N,p)) \text{ for all } (N,p) \in \mathcal{P}\} \\
= \;& \{G : G^d \in E_F(\mathcal{P},\varphi)\}.
\end{aligned}$$

Since $G((N,p),\varphi^d(N,p)) = 1$, $G^d((N,p),\varphi(N,p)) = 1$. Then, for all $(N,p) \in \mathcal{P}$ and all $G^d \in E_F(\mathcal{P},\varphi)$, $G^d((N,p),\varphi(N,p)) = 1$, which implies that φ satisfies an axiom $E_F(\mathcal{P},\varphi)$. By the same argument, we obtain the opposite implication. ■

Next, the following theorem shows that one can axiomatize the *dual* of a rule by taking the *dual* of the axioms involved in an axiomatization of the original rule.

Theorem 2 *(Axiomatization Theorem for dual rules) Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and propositional functions $F_l \in \mathcal{F}$ $(l = 1, 2, \cdots, k)$, if a rule φ on $\mathcal{P}$ is axiomatized by axioms $E_{F_l}(\mathcal{P}, \varphi)$ $(l = 1, 2, \cdots, k)$, then the dual rule φ^d is axiomatized by the dual axioms $E^d_{F_l}(\mathcal{P}, \varphi^d)$ $(l = 1, 2, \cdots, k)$.*

Proof. Let φ be a rule on $\mathcal{P}$, satisfying $E_{F_l}(\mathcal{P}, \varphi)$ $(l = 1, 2, \cdots, k)$. By Theorem 1, φ^d satisfies $E^d_{F_l}(\mathcal{P}, \varphi^d)$ $(l = 1, 2, \cdots, k)$. Suppose that φ is the unique rule on $\mathcal{P}$, satisfying $E_{F_l}(\mathcal{P}, \varphi)$ $(l = 1, 2, \cdots, k)$, and $\tilde{\varphi}$ is any rule on $\mathcal{P}$, satisfying $E^d_{F_l}(\mathcal{P}, \varphi^d)$ $(l = 1, 2, \cdots, k)$. Since $\tilde{\varphi} = (\tilde{\varphi}^d)^d$, $(\tilde{\varphi}^d)^d$ satisfies $E^d_{F_l}(\mathcal{P}, \varphi^d)$ $(l = 1, 2, \cdots, k)$. Again, by Theorem 1, $\tilde{\varphi}^d$ satisfies $E_{F_l}(\mathcal{P}, \varphi)$ $(l = 1, 2, \cdots, k)$. Hence, $\tilde{\varphi}^d = \varphi$, or equivalently, $\tilde{\varphi} = \varphi^d$, which implies that $\tilde{\varphi}$ is unique. ■

3.2.3 Anti-dual axioms, and anti-dual axiomatizations

As in the case of dual axioms for rules, we introduce the mathematical structure of anti-dual axioms for rules.

Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and a propositional function $F \in \mathcal{F}$, the **anti-dual of $\boldsymbol{F}$**, denoted F^{ad}, is defined by setting, for all $(N, p) \in \mathcal{P}$,

$$F^{ad}((N, p), \varphi(N, p)) \equiv F((N, p), \varphi^{ad}(N, p)),$$

where $\varphi^{ad}(N, p) = -\phi \circ v^{ad}_P(N, p)$ for all $(N, p) \in \mathcal{P}$ and all $v_P \in \mathcal{V}_P$.

Given a propositional function $\bar{F} \in \mathcal{F}$, an **anti-dual axiom for a rule $\boldsymbol{\varphi}$ with respect to $\bar{F}$**, denoted $E^{ad}_{\bar{F}}(\mathcal{P}, \varphi)$, is defined by setting, for all $(N, p) \in \mathcal{P}$,

$$E^{ad}_{\bar{F}}(\mathcal{P}, \varphi) \equiv E_{\bar{F}^{ad}}(\mathcal{P}, \varphi).$$

The following theorem is the anti-dual version of Theorem 1. The proof is the same as that of Theorem 1. We omit it.

Theorem 3 *(Existence Theorem of anti-dual axioms for rules) Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and a propositional function $F \in \mathcal{F}$, a rule φ satisfies an axiom $E_F(\mathcal{P}, \varphi)$ if and only if the anti-dual rule φ^{ad} satisfies the anti-dual axiom $E^{ad}_F(\mathcal{P}, \varphi^{ad})$.*

Next, the following theorem is the anti-dual version of Theorem 2. The proof is the same as that of Theorem 2. We omit it.

Theorem 4 *(Axiomatization Theorem for anti-dual rules) Given a class $\mathcal{P}$, a class $\mathcal{V}_P$, a solution ϕ on $\mathcal{V}_P$, and propositional functions $F_l \in \mathcal{F}$ $(l = 1, 2, \cdots, k)$, if a rule φ on $\mathcal{P}$ is axiomatized by axioms $E_{F_l}(\mathcal{P}, \varphi)$ $(l = 1, 2, \cdots, k)$, then the anti-dual rule φ^{ad} is axiomatized by the anti-dual axioms $E^{ad}_{F_l}(\mathcal{P}, \varphi^{ad})$ $(l = 1, 2, \cdots, k)$.*

An economic application of Theorem 2 is as follows: Suppose that we have an axiomatization of rule φ for allocation problems and its dual is rule φ^d for distinct allocation problems. Suppose that in the existing literature no axiomatization of rule φ^d is investigated. Then just by identifying the dual of each axiom involved in axiomatization of rule φ, we obtain an axiomatization of rule φ^d. An economic application of Theorem 4 is the same as in the case of Theorem 2.

In the following sections, we show examples of these applications mentioned above.

4 Duality and anti-duality approach to analysis of allocation problems

In this section, by using the notion of *duality*, we axiomatize the Shapley rule for bidding ring problems. Notice that one can axiomatize the Shapley rule for the problems by using the notion of *anti-duality.* For simplicity of our analysis, we take the *duality* approach here.

4.1 Airport problems, and bidding ring problems

There is a set of airlines for whom an airstrip they will jointly use is to be built. Each airline owns one type of aircraft. Airlines have different needs for airstrips, since they own different types of aircraft. An airstrip needed to accommodate the largest aircraft is to be built. The problem is to determine how to share the cost of the airstrip between the airlines (Littlechild and Owen 1973).

An **airport problem** is a pair (N, c), where $N \in \mathcal{N}$ is the set of airlines and $c = (c_i)_{i \in N}$ is the profile of cost parameters, namely c_i is the construction cost of the airstrip for airline i. We assume that the cost is increasing in the length of the airstrip. For simplicity, we assume that $c_n \geq c_{n-1} \geq \cdots \geq c_1 > 0$. Let $\mathcal{C}$ be the class of all airport problems on $\mathcal{N}$.

Given $(N, c) \in \mathcal{C}$, the **airport game** is defined by setting, for all $S \subseteq N$,

$$c_A(N, c)(S) \equiv \max_{i \in S} c_i.$$

For all $S \in 2^N$, $c_A(N, c)(S)$ represents the cost of the airstrip needed to accommodate the members of coalition S. It is equal to the cost of the airstrip needed to accommodate the member of the coalition whose cost parameter is the largest.

Let $\mathcal{C}_A$ be the class of all airport games. Given $(N, c) \in \mathcal{C}$, an allocation for (N, c) is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = \max_N c_i$ (which is equal to

c_n). Let $X(N,c)$ be the set of allocations for (N,c). A **solution for airport games** is a mapping $\phi_A : \mathcal{C}_A \to \mathbb{R}^N$ that associates with each airport game $c_A(N,c)$ in the domain an allocation in $X(N,c)$. We refer to the composite mapping $\varphi_A \equiv \phi_A \circ c_A$ as a **rule for airport problems**. The **Shapley rule for airport problems** is defined by $\varphi_A^{Sh} \equiv Sh \circ c_A$. The **nucleolus rule for airport problems** is defined by $\varphi_A^{Nu} \equiv Nu \circ -c_A$, since airport problems are cost problems and the nucleolus is defined under the situation of profit games.

An **English auction** is an oral auction in which an auctioneer initially sets a bid at a seller's reservation price and then gradually increases the price until only one bidder remains active. There is a set of buyers in a *single-object English auction*. There is no asymmetry of information between the buyers; that is, each buyer has information on the valuations of all buyers for the object. The valuation of each buyer is positive, and all valuations are different. The reservation price is zero. A bidding ring is formed by all buyers. The bidding ring wins the auction by making the buyer whose valuation is the highest the sole bidder. The benefit of the ring members' strategy is equal to the valuation of this buyer. The problem for the members in the ring is to determine how to share the benefit of their strategy (Graham et al. 1990).

A **bidding ring problem** is a pair (N,c), where $N \in \mathcal{N}$ is the set of buyers and $c = (c_i)_{i \in N}$ is the profile of valuations for a single object, c_i being the valuation of buyer i. For simplicity, we assume that $c_n \geq c_{n-1} \geq \cdots \geq c_1 > 0$. Let $\mathcal{C}$ be the class of all bidding ring problems on $\mathcal{N}$.

Given $(N,c) \in \mathcal{C}$, the **bidding ring game** is defined by setting, for all $S \subseteq N$,

$$v_B(N,c)(S) = \begin{cases} c_n - \max_{j \notin S} c_j & \text{if } S \ni n \\ 0 & \text{if } S \not\ni n, \end{cases}$$

where $\max_{j \notin N} c_j \equiv 0$. The intuition is as follows: First, under the English auction rule, it is a dominant strategy for each bidder to remain active until bidding reaches his valuation. Second, any coalition including buyer n can win the auction, and achieve the net benefit $c_n - \max_{j \notin S} c_j$ by making buyer n the sole bidder in the coalition and his bidding c_n. Finally, no coalition that does not include buyer n wins the auction, and hence its net benefit is 0.

Let $\mathcal{V}_B$ be the class of all bidding ring games. Given $(N,c) \in \mathcal{C}$, an allocation for (N,c) is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = c_n$. Let $X(N,c)$ be the set of allocations for (N,c) . A **solution for bidding ring games** is a mapping $\phi_B : \mathcal{V}_B \to \mathbb{R}^N$ that associates with each bidding ring game $v_B(N,c)$ in the domain an allocation in $X(N,c)$. We refer to the composite mapping $\varphi_B \equiv \phi_B \circ v_B$ as a **rule for bidding ring problems**. The **Shapley rule for bidding ring problems** is defined by $\varphi_B^{Sh} \equiv Sh \circ v_B$. The **nucleolus rule for bidding ring problems** is defined by $\varphi_B^{Nu} \equiv Nu \circ v_B$.

Remark 1 *The following assertions hold:*

(i) The class $\mathcal{C}_A$ of airport games and the class $\mathcal{V}_B$ of bidding ring games are dual.

(ii) The Shapley value of airport games on the domain $\mathcal{C}_A$ coincides with that of bidding ring games on the domain $\mathcal{V}_B$.

(iii) The nucleolus multiplied by -1 of airport games on the domain $\mathcal{C}_A$ coincides with that of bidding ring games on the domain $\mathcal{V}_B$.

4.2 Duality approach to bidding ring problems

In the existing literature, the Shapley rule for bidding ring problems has not been axiomatized. Just by identifying the *dual* of each axiom involved in an axiomatization of φ_A^{Sh}, we obtain an axiomatization of φ_B^{Sh}.

Let us consider the *dual* of each axiom involved in the axiomatization of the Shapley rule for airport problems (Chun et al. 2012).

First, we consider the following property. Each airline i has the right to use at least the airstrip to accommodate the airline i. It says that each airline i should pay at least an equal share of c_i.

Equal share lower bound for airport problems (ESL for airport problems): For all $(N, c) \in \mathcal{C}$ and all $i \in N$,

$$\varphi_{A[i]}(N, c) \geq \frac{c_i}{n}.$$

The following property says that each buyer $i \in N$ should gain at least an equal share of his valuation. It is *self-dual.*

Equal share lower bound for bidding ring problems (ESL for bidding ring problems): For all $(N, c) \in \mathcal{C}$ and all $i \in N$,

$$\varphi_{B[i]}(N, c) \geq \frac{c_i}{n}.$$

Next, we consider the following property for airport problems. It requires that if the cost of an airline increases, then all the other airlines should pay at most as much as they did initially.

Individual monotonicity for airport problems (IM for airport problems): Fix an arbitrary $N \in \mathcal{N}$. For all $(N, c) \in \mathcal{C}$, all $(N, c') \in \mathcal{C}$, and all $i \in N$, if $c'_i > c_i$, and for all $j \in N \backslash \{i\}$, $c'_j = c_j$, then for all $j \in N \backslash \{i\}$,

$$\varphi_{A[j]}(N, c') \leq \varphi_{A[j]}(N, c).$$

The following property says that if the valuation of a buyer increases, then all the other buyers should share at most as much as they did initially. It is *self-dual.*

Individual monotonicity for bidding ring problems (IM for bidding ring problems): Fix an arbitrary $N \in \mathcal{N}$. For all $(N, c) \in \mathcal{C}$, all $(N, c') \in \mathcal{C}$, and all $i \in N$, if $c'_i > c_i$, and for all $j \in N \backslash \{i\}$, $c'_j = c_j$, then for all $j \in N \backslash \{i\}$,

$$\varphi_{B[j]}(N, c') \leq \varphi_{B[j]}(N, c).$$

Our final property for airport problems says that if a new airline arrives, then all airlines whose costs are more than the cost of the new airline should be affected equally.

Population fairness for airport problems (PF for airport problems): For all $N \in \mathcal{N}$, all $(N, c) \in \mathcal{C}$, all $i \in \mathcal{I} \backslash N$, all $j, k \in N$ such that $\min\{c_j, c_k\} > c_i$,

$$\varphi_{A[j]}(N \cup \{i\}, c_{N \cup \{i\}}) - \varphi_{A[j]}(N, c) = \varphi_{A[k]}(N \cup \{i\}, c_{N \cup \{i\}}) - \varphi_{A[k]}(N, c).$$

The following property says that if a new buyer arrives, then all buyers whose evaluations are more than the valuation of the new buyer should be affected equally. It is *self-dual.*

Population fairness for bidding ring problems (PF for bidding ring problems): For all $N \in \mathcal{N}$, all $(N, c) \in \mathcal{C}$, all $i \in \mathcal{I} \backslash N$, all $j, k \in N$ such that $\min\{c_j, c_k\} > c_i$,

$$\varphi_{B[j]}(N \cup \{i\}, c_{N \cup \{i\}}) - \varphi_{B[j]}(N, c) = \varphi_{B[k]}(N \cup \{i\}, c_{N \cup \{i\}}) - \varphi_{B[k]}(N, c).$$

Thus, we obtain the following axiomatization of solution φ_B^{Sh} that is *self-dual* of the axiomatization of solution φ_A^{Sh}.

Theorem A *(Chun et al. 2012) For airport problems, the Shapley rule is the only rule satisfying the equal share lower bound, individual monotonicity, and population fairness.*

Theorem 5 *(Self-dual of Theorem A) For bidding ring problems, the Shapley rule is the only rule satisfying the equal share lower bound, individual monotonicity, and population fairness.*

Proof. We consider the following steps.

Step 1: ESL for bidding ring problems is the self-dual of ESL for airport problems.

ESL for airport problems is expressed by

$$F((N,c), Sh \circ c_A(N,c)) = 1 \Leftrightarrow [Sh \circ c_A(N,c)]_i \geq \frac{c_A(i)}{n}.$$

Then, this formula can be rewritten by the formulas successively.

$$\begin{aligned}
F((N,c), Sh \circ (c_A^d)^d(N,c)) &= 1 \Leftrightarrow [Sh \circ (c_A^d)^d(N,c)]_i \geq \frac{(c_A^d)^d(i)}{n};\\
F((N,c), Sh \circ (v_B)^d(N,c)) &= 1 \Leftrightarrow [Sh \circ (v_B)^d(N,c)]_i \geq \frac{(v_B)^d(i)}{n};\\
F^d((N,c), Sh \circ (v_B)(N,c)) &= 1 \Leftrightarrow [Sh \circ (v_B)^d(N,c)]_i \geq \frac{(v_B)^d(i)}{n},
\end{aligned}$$

where

$$\begin{aligned}
(v_B)^d(i) &= v_B(N) - v_B(N\backslash\{i\})\\
&= c_n - (c_n - c_i)\\
&= c_i.
\end{aligned}$$

By the self duality that $Sh \circ (v_B)^d(N,c) = Sh \circ (v_B)(N,c)$,

$$F^d((N,c), Sh \circ (v_B)(N,c)) = 1 \Leftrightarrow [Sh \circ (v_B)(N,c)]_i \geq \frac{c_i}{n},$$

a desired claim.

Step 2: IM for bidding ring problems is the self-dual of IM for airport problems.

For all $(N,c) \in \mathcal{C}$, all $(N,c') \in \mathcal{C}$, all $i \in N$, and for all $j \in N\backslash\{i\}$, let $c'_A(i) > c_A(i)$, and $c'_A(j) = c_A(j)$. IM for airport problems is expressed by

$$F((N,c), Sh \circ c_A(N,c)) = 1 \Leftrightarrow \forall j \in N\backslash\{i\} : [Sh \circ c_A(N,c')]_j \leq [Sh \circ c_A(N,c)]_j$$

Then, this formula can be rewritten by the formulas successively.

$$\begin{aligned}
F((N,c), Sh \circ (c_A^d)^d(N,c)) &= 1 \Leftrightarrow \forall j \in N\backslash\{i\} : [Sh \circ (c_A^d)^d(N,c')]_j \leq [Sh \circ (c_A^d)^d(N,c)]_j;\\
F((N,c), Sh \circ (v_B)^d(N,c)) &= 1 \Leftrightarrow \forall j \in N\backslash\{i\} : [Sh \circ (v_B)^d(N,c')]_j \leq [Sh \circ (v_B)^d(N,c)]_j;\\
F^d((N,c), Sh \circ (v_B)(N,c)) &= 1 \Leftrightarrow \forall j \in N\backslash\{i\} : [Sh \circ (v_B)(N,c')]_j \leq [Sh \circ (v_B)(N,c)]_j.
\end{aligned}$$

For all $(N,c) \in \mathcal{C}$, all $(N,c') \in \mathcal{C}$, all $i \in N$, all $j \in N\backslash\{i\}$,

$$((c'_A)^d)^d(i) > (c_A^d)^d(i),\ ((c'_A)^d)^d(j) = (c_A^d)^d(j).$$

This implies that for all $(N, c) \in \mathcal{C}$, all $(N, c') \in \mathcal{C}$, all $i \in N$, all $j \in N \backslash \{i\}$,

$$(v'_B)^d(i) > (v_B)^d(i),\ (v'_B)^d(j) = (v_B)^d(j),$$

which implies that $c'_i > c_i$ and $c'_j = c_j$ since $(v_B)^d(k) = c_k$ for all $k \in N$. Hence we obtain the desired claim.

Step 3: PF for bidding ring problems is the self-dual of PF for airport problems.

PF for airport problems is expressed by

$F((N, c), Sh \circ c_A(N, c)) = 1 \Leftrightarrow$ for all $N \in \mathcal{N}$, all $(N, c) \in \mathcal{C}$, all $i \in \mathcal{I} \backslash N$, all $j, k \in N$ such that $\min\{c_A(j), c_A(k)\} > c_A(i)$,

$$Sh \circ c_{A[j]}(N \cup \{i\}, c_{N \cup \{i\}}) - Sh \circ c_{A[j]}(N, c) = Sh \circ c_{A[k]}(N \cup \{i\}, c_{N \cup \{i\}}) - Sh \circ c_{A[k]}(N, c).$$

Then, this can be written by the formulas successively.

$F((N, c), Sh \circ (c_A^d)^d(N, c)) = 1$

$\Leftrightarrow$ for all $N \in \mathcal{N}$, all $(N, c) \in \mathcal{C}$, all $i \in \mathcal{I} \backslash N$, all $j, k \in N$ such that $\min\{(c_A^d)^d(j), (c_A^d)^d(k)\} > (c_A^d)^d(i)$,

$$Sh \circ (c_A^d)^d_{[j]}(N \cup \{i\}, c_{N \cup \{i\}}) - Sh \circ (c_A^d)^d_{[j]}(N, c) = Sh \circ (c_A^d)^d_{[k]}(N \cup \{i\}, c_{N \cup \{i\}}) - Sh \circ (c_A^d)^d_{[k]}(N, c).$$

$F((N, c), Sh \circ (v_B)^d(N, c)) = 1$

$\Leftrightarrow$ for all $N \in \mathcal{N}$, all $(N, c) \in \mathcal{C}$, all $i \in \mathcal{I} \backslash N$, all $j, k \in N$ such that $\min\{(v_B)^d(j), (v_B)^d(k)\} > (v_B)^d(i)$,

$$Sh \circ (v_B)^d_{[j]}(N \cup \{i\}, c_{N \cup \{i\}}) - Sh \circ (v_B)^d_{[j]}(N, c) = Sh \circ (v_B)^d_{[k]}(N \cup \{i\}, c_{N \cup \{i\}}) - Sh \circ (v_B)^d_{[k]}(N, c).$$

$F^d((N, c), Sh \circ (v_B)(N, c)) = 1$

$\Leftrightarrow$ for all $N \in \mathcal{N}$, all $(N, c) \in \mathcal{C}$, all $i \in \mathcal{I} \backslash N$, all $j, k \in N$ such that $\min\{c_j, c_k\} > c_i$,

$$Sh \circ (v_B)_{[j]}(N \cup \{i\}, c_{N \cup \{i\}}) - Sh \circ (v_B)_{[j]}(N, c) = Sh \circ (v_B)_{[k]}(N \cup \{i\}, c_{N \cup \{i\}}) - Sh \circ (v_B)_{[k]}, (N, c),$$

a desired claim.

Step 4: By step1 to step 3 together with Theorem 2, we obtain the desired theorem. ■

5 Anti-duality approach to bidding ring problems

In the existing literature, the nucleolus rule for bidding ring problems has not been axiomatized. Just by identifying the *anti-dual* of each axiom involved in an axiomatization of φ_A^{Nu}, we obtain an axiomatization of φ_B^{Nu}.

Let us consider the *anti-dual* of each axiom involved in the axiomatization of the nucleolus rule for airport problems (Hwang and Yeh 2012).

First, we consider the following property. It says that airlines with equal costs should contribute equal amounts.

Equal treatment of equals for airport problems (ETE for airport problems): For each $(N, c) \in \mathcal{C}$ and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $\varphi_i^A(N, c) = \varphi_j^A(N, c)$.

The anti-dual of this axioms is *self-anti-dual.* Since the proof is the same way as in Theorem 5, we omit it. This property say that buyers with equal valuations should gain equal amounts.

Equal treatment of equals for bidding ring problems (ETE for bidding ring problems): For each $(N, c) \in \mathcal{C}$ and each pair $\{i, j\} \subseteq N$, if $c_i = c_j$, then $\varphi_i^B(N, c) = \varphi_j^B(N, c)$.

Next we consider the following property. It says that if the cost of an airline with the highest cost increases by δ, then all other airlines should contribute the same amounts as they did initially.

Last-agent additivity for airport problems (Last-agent add for airport problems): For each pair $\{(N, c), (N, c')\}$ of elements of $\mathcal{C}$ and each $\delta \in \mathbb{R}_+$, if $c'_n = c_n + \delta$ and for each $j \in N \backslash \{n\}$ $c'_j = c_j$, then $\varphi_n^A(N, c') = \varphi_n^A(N, c) + \delta$ and for each $j \in N \backslash \{n\}$, $\varphi_j^A(N, c') = \varphi_j^A(N, c)$.

The anti-dual of this axioms is *self-anti-dual.* Since the proof is the same way as in Theorem 5, we omit it. This property says that if the valuation of a buyer with the highest valuation increases by δ, then all other buyers should gain the same amounts as they did initially.

Last-agent additivity for bidding ring problems (Last-agent add for bidding ring problems): For each pair $\{(N, c), (N, c')\}$ of elements of $\mathcal{C}$ and each $\delta \in \mathbb{R}_+$, if $c'_n = c_n + \delta$ and for each $j \in N \backslash \{n\}$ $c'_j = c_j$, then $\varphi_n^B(N, c') = \varphi_n^B(N, c) + \delta$ and for each $j \in N \backslash \{n\}$, $\varphi_j^B(N, c') = \varphi_j^B(N, c)$.

Finally, we consider the following property. Imagine that airline i pays its contribution x_i and leaves. The remaining airlines' costs are revised as follows: (i) for each airline j whose cost is lower than airline i's cost, its revised cost is the minimum of c_j and $c_i - x_i$, and (ii) for each airline j whose cost is higher than airline i's cost, its revised cost is $c_j - x_i$. Thus, the reduced problem consists of the set of the remaining airlines and the cost parameters revised. For the details of a justification of the reduced problem, see Hwang and Yeh

(2012). The airport consistency says that for the reduced problem the outcome chosen by a rule should be invariant. Using the propositional functions, we state the property as follows.

Airport consistency: For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, $F((N, c), Nu \circ -c_A(N, c)) = 1$ iff for all $i \in N$ and $x = Nu \circ -c_A(N, c)$,

$$(N \backslash \{i\}, c^x_{N\backslash\{i\}}) \in \mathcal{C} \text{ and } x_{N\backslash\{i\}} = Nu \circ -c^x_{N\backslash\{i\}},$$

where

(i) $(c^x_{N\backslash\{i\}})_{[j]} = \min\{c_j, c_i - x_i\}$ for each $j \in N\backslash\{i\}$ such that $j < i$;
(ii) $(c^x_{N\backslash\{i\}})_{[j]} = c_j - x_i$ for each $j \in N\backslash\{i\}$ such that $j > i$.

Notice that the cost parameters revised are derived from "Davis-Maschler consistency" (Davis and Maschler 1965). In fact, thanks to Davis-Maschler consistency, for each $j \in N\backslash\{i\}$, we can set the cost parameter revised $(c^x_{N\backslash\{i\}})_{[j]}$ as $(c^x_{N\backslash\{i\}})(j)$ such that

$$\begin{aligned}(c^x_{N\backslash\{i\}})(j) &= \min_{Q \subseteq \{i\}} [c_A(\{j \cup Q\}) - x(Q)] \\ &= \begin{cases} \min\{c_j, c_i - x_i\} & \text{if } j < i \\ c_j - x_i & \text{otherwise.} \end{cases}\end{aligned}$$

We consider the anti-dual of airport consistency. For notational convenience, let $N' \equiv N\backslash\{i\}$. By Theorem 4, the formula of the airport consistency can be written by the formulas (1), (2), and (3), successively.

(1) For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, $F((N, c), Nu \circ -c_A(N, c)) = 1$ iff for all $i \in N$ and $x = Nu \circ -c_A(N, c)$, $(N', \{c^x_{A,N'}(j)\}_{j\in N'}) \in \mathcal{C}$ and $x_{N'} = Nu \circ -c^x_{A,N'}$.

(2) For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, $F((N, c), -Nu \circ (-v_B)^{ad}(N, c)) = 1$ iff for all $i \in N$ and $x = -Nu \circ (-v_B)^{ad}(N, c)$, $(N', \{((-v_B)^{-x}_{N'})^{ad}(j)\}_{j\in N'}) \in \mathcal{C}$ and $-x_{N'} = -Nu \circ ((-v_B)^{-x}_{N'})^{ad}$.

(3) For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, $F^{ad}((N, c), Nu \circ v_B(N, c)) = 1$ iff for all $i \in N$ and $x = Nu \circ v_B(N, c)$, $(N', \{v_{B,N'}^{x}(j)\}_{j\in N'}) \in \mathcal{C}$ and $x_{N'} = Nu \circ v_{B,N'}^{x}$.

We must compute the value of $((-v_B)^{-x}_{N'})^{ad}(j)$. First, for each $S \subsetneq N'$ the number of $(c^x_{N'})(S)$ is given by

$$\begin{aligned}(c^x_{N'})(S) &= \min_{Q \subseteq \{i\}} [c_A(S \cup Q) - x(Q)] = \min_{Q \subseteq \{i\}} [c^d_A(S \cup Q) - x(Q)] \\ &= \min_{Q \subseteq \{i\}} [v_B(S \cup Q) - x(Q)]\end{aligned}$$

Then, for each $S \subsetneq N'$ the number of $(-v_B)^{-x}_{N'}(S)$ is given by

$$\begin{aligned}(-v_B)^{-x}_{N'}(S) &= \min_{Q\subseteq\{i\}} \left[-v_B(S\cup Q) + x(Q)\right] \\ &= \min_{Q\subseteq\{i\}} \left[-(c_n - \max_{k\in N\setminus(S\cup Q)} c_k) + x(Q)\right] \\ &= -\max_{Q\subseteq\{i\}} \left[c_n - \max_{k\in N\setminus(S\cup Q)} c_k - x(Q)\right],\end{aligned}$$

since Davis-Maschler reduces games are *self-anti-duals* (Oishi et al. 2016).

For each $S \subsetneq N'$, let $w(S) \equiv ((-v_B)^{-x}_{N'})^{ad}(S)$. We have that $w(N') = c_n - x_i$ and for each $j \in N'$

$$\begin{aligned}w(N'\setminus\{j\}) &= -\max_{Q\subseteq\{i\}} \left[c_n - \max_{k\in N\setminus((N'\setminus j)\cup Q)} c_k - x(Q)\right] \\ &= -\max\{c_n - c_j - x_i, c_n - \max\{c_i, c_j\}\}.\end{aligned}$$

For each $j \in N'$, since $((-v_B)^{-x}_{N'})^{ad}(j) = -w(N') + w(N'\setminus\{j\})$,

$$((-v_B)^{-x}_{N'})^{ad}(j) = (c_n - x_i) - \max\{c_n - c_j - x_i, c_n - \max\{c_i, c_j\}\},$$

which reduces to

$$((-v_B)^{-x}_{N'})^{ad}(j) = \begin{cases}\min\{c_j, c_i - x_i\} & \text{if } j < i \\ c_j - x_i & \text{otherwise.}\end{cases}$$

Thanks to the approach proposed, we emphasize that an appropriate interpretation of a consistency property in the context of bidding ring problems is possible although the airport consistency is *self-anti-dual.* Again, let consider the valuation revised $(c_n - x_i) - \max\{c_n - c_j - x_i, c_n - \max\{c_i, c_j\}\}$. First, we can regard the valuation of each buyer as his contribution to the grand bidding ring N. The number of $c_n - x_i$ is the benefit of N' when N' pays x_i to buyer i as a reward for his cooperation. The number of $\max\{c_n - c_j - x_i, c_n - \max\{c_i, c_j\}\}$ is the possibly highest benefit of $N'\setminus\{j\}$ when buyer j competes with the members of a bidding ring $N'\setminus\{j\}$. Under this situation, we take into consideration that for the bidding ring N' buyer i behaves cooperatively or non-cooperatively. Therefore, the valuation revised is interpreted as "buyer j's contribution to the bidding ring N'". As a result, we obtain the following interpretation. Imagine that the grand bidding ring N pays x_i to buyer i as a reward for his cooperation, and buyer i leaves. The remaining buyers' contributions are revised as contributions to the bidding ring N'. Thus, the reduced problem consists of the set of the remaining buyers and the contribution parameters revised. The bidding ring consistency says that for the reduced problem the outcome chosen by a rule should be invariant.

Bidding ring consistency: For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, $F((N, c), Nu \circ v_B(N, c)) = 1$ iff for all $i \in N$ and $x = Nu \circ v_B(N, c)$,

$$(N\backslash\{i\}, r^x_{N\backslash\{i\}}) \in \mathcal{C} \text{ and } x_{N\backslash\{i\}} = Nu \circ v_B(N\backslash\{i\}, r^x_{N\backslash\{i\}}),$$

where for each $j \in N\backslash\{i\}$ $(r^x_{N\backslash\{i\}})_{[j]} = (c_n - x_i) - \max\{c_n - c_j - x_i, c_n - \max\{c_i, c_j\}\}$.

In summary, we obtain an axiomatization of the nucleolus rule for bidding ring problems.

Theorem B *(Hwang and Yeh 2012) For airport problems, the nucleolus rule is the only rule satisfying equal treatment of equals, last-agent additivity, and airport consistency.*

Theorem 6 *(Self-dual of Theorem B, but a different form) For bidding ring problems, the nucleolus rule is the only rule satisfying equal treatment of equals, last-agent additivity, and bidding ring consistency.*

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