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Optimal employment in frictional business cycles and intertemporal discontinuity of production and internal prices

Koji Yokota

(Meisei University)

Hodokubo 2-1-1, Hino, Tokyo 191-8506

School of Economics, Meisei University

Phone:+81-(0)42-591-9479 Fax: +81-(0)42-599-3024

URL: https://keizai.meisei-u.ac.jp/econ/

E-mail: keizai@econ.meisei-u.ac.jp

Optimal employment in frictional business cycles and intertemporal discontinuity of production and internal prices*

By Koji Yokota

Labor market with nonbinding contract has a particular price mechanism that depends on bargaining. Can there be a case where no autonomous mechanism is equipped that always brings the economy back to the long-term steady state? This paper shows that existence of convex hiring cost together with differentiated goods market allows multiple prolonged inefficient paths out of long-term steady states. The optimal behavior of firms in those inefficient paths has peculiar characteristics expecially at the margin of firing phases.
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I. Introduction

The present paper studies the optimal behavior of the firm facing labor friction characterized by convex hiring cost and sufficiently differentiated output good. Even though there are conflicting arguments on empirical applicability of convex hiring cost, from the point of view that hiring cost is a variation of adjustment cost, it should be natural to assume it from the theoretical perspective.[1] The convexity in hiring cost makes the labor adjustment process time-consuming, which in turn determines the level of income and demand for output at every moment while the adjustment process is still going on. When output goods are sufficiently differentiated, for such an adjustment process to be fulfilled, expectation on the social choice of the equilibrium path must be shared uninterruptedly among firms that the economy ultimately reaches to the long-term equilibrium. If expectation is not well-coordinated, the investment in labor by a single firm according to the efficient path will cause a loss, since any discount in output prices will not retrieve the cost of investment. It raises multiple inefficient paths that depend on the degree of coordination. Yashiv (2006, 2007) analyzed the efficient equilibrium

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[1] Yashiv (2000) and Blatter, Muehlemann and Schenker (2012) support convexity. On the other hand, Abowd and Kramarz (2003) and Kramarz and Michaud (2010) finds concavity for long-term workers in France. However, the hiring cost in the latter is defined to be directly observable ones which differs from our setup that the cost arises from the wage payment to workers internally allocated to the hiring sector and arises from decreasing returns in both production and hiring sectors.

in an economy with convex hiring cost. The present paper extends them to those inefficient cases. In the sense that expectation is the main driving force of business cycles, the model shares the spirit with news-driven business cycle models by Beaudry and Portier (2004) and others. However, in our model, the source of expectation is not limited to technological news on productivity fluctuation, but it is more generalized shared view about future equilibrium characterized by nature of beauty contest.

A firm operating in an economy with labor friction has an intertemporal horizon. However, a model of a going concern with no firing cost easily falls in the category of singular control problems. They are the cases in which Hamiltonian becomes independent from some control variables and the (ordinal procedure) of the maximum principle does not allow to find the optimal control value.[2] It occurs when the switching function becomes zero. Even though there were cases that singularity is artificially evaded in early times as Johnson and Gibson (1963) pointed out, obviously it is not guaranteed that those singularity should be "pathological". The model studied in this paper shows that singularity is indeed not only unexcludable but comprises important part of business cycles. Namely, the firing phase is singular. Moreover, since the singular control is derived from the resulting state constraint on the boundary, the optimal control in the firing phase is determined only in a derivative form, leaving determination of the initial condition out of the firing phase. It brings intertemporal discontinuity in costate variables and thus in the output level. This is a similar property to optimal control problems with state variable inequality constraints (SVICs), which is basically brought by the truncatability of the problem into subperiods. It is known that the occurrence time of costate discontinuity in state-constrained problems is generally indeterminate between entering and leaving times.[3] It will be proved that our setup shows the discontinuity in both times.

The properties of the model with no firing cost gives a good implication for a model with firing cost. It tells us that firing is optimal to visit in a chunk on the first day of the firing phase in general. With firing cost, such a discontinuous behavior is too costly. The firm is better to keep idle employment as far as the cost to hold it measured in its absolute value (which is negative) is less than the firing cost. Thus, it brings labor hoarding.

Section II explains the model. Section III analyzes steady states both with unbounded and bounded demand constraints. Section V and VI study out of steady states. Section V examines firing phase and Section VI examines the entering and exiting from those phases. Section VII examines implication to labor hoarding. Section VIII studies how labor hoarding behavior changes when there is convex firing cost. Section IX shows that linear firing cost mixes results of the basic model and that of Section VIII. Section X concludes.

[2]Rozonoer (1959).

[3]This is true for many of problems. Hartl, Sethi and Vickson (1995) points out a case in which the discontinuity occurs within the boundary intervals.

II. The model with no firing regulation: a singular control problem

Consider an economy with differentiated output goods produced by labor. The labor market is frictional characterized by convex hiring cost, which allows inefficient paths of the economy. Goods are perishable and normal. The goods market is characterized by infinitely many producers, however, reflecting the differentiation of output goods, the market for each output good is monopolistic.[4]

A. Households

The complexity of the demand side is minimized. There are infinitely many differentiated output goods. They belong to the same category so that they are all symmetric in the utility function. Households are illustrated as a representative consumer. She or he is endowed with the ownership of all firms with equal weights as non-transferrable equities. The goods are perishable so that there is no room for saving by goods. The problem for the representative consumer degenerates to instantaneous utility maximization. At any moment, he or she can be in a state of employed (state-0) or unemployed (state-1). The value function of state j as of time t is denoted by $u_t^{(j)}$, the maximand of which is an additively seperable utility function.

$$u_t^{(j)} = \max_{\{c\}} \int_{i\in\Omega} u(q_i)\,\mathrm{d}i \qquad \text{s.t.} \quad \int_{i\in\Omega} p_i\, q_i\, \mathrm{d}i \le w^{(j)} + \pi$$

where Ω is the set of firms the measure of which is fixed and normalized to one, $u \in C^2$ belongs to CRRA class satisfying $u' > 0$ and $u'' < 0$, q_i is the demand for i-th output good, p_i is its price, $w^{(j)}$ is income in state j such that $w^{(0)} > 0$ and $w^{(1)} = 0$ and π is profits of own firms. Since firms are homogeneous, output price is normalized to $p_i = 1$ for any i in equilibrium so that $q_i = w^{(j)} + \pi$ holds for any i. Simple as the outcome is, the demand structure mainly affects firms in off-equilibrium paths, which is analyzed in Section II.C.

B. Firms

Firms are non-atomic in the output market having aggregate measure one. They engage in two activities, production of goods and hiring of workers. Both activities require two inputs, labor and land where the latter is assumed to be fixed throughout the analysis. The land usage in the production and hiring sectors are fixed and total real land cost is $c \ge 0$. The production function of goods is

[4]Firms are monopolistic in the sense that they are allowed to manipulate output prices in off-equilibrium context. However, nominal price manipulation will not be undertaken so that the same level of "normal" profits are shared among firms.

therefore denoted by $f(\hat{l})$ where fixed land input is implicitly embedded in the functional form of f, $\hat{l} \geq 0$ is the measure of employment in the production sector and f shows decreasing returns. More precisely, it is characterized by $f' > 0$, $f'' < 0$, $f(0) = 0$ and Inada conditions. Another activity, hiring, requires expense of internal resources to get hiring results. It is expressed by another production function $g(\tilde{l};\theta)$ or simply $g(\tilde{l})$ in harmless places where land usage is implicit again, $\tilde{l} \geq 0$ is the measure of employment in the hiring sector and θ is a parameter which represents vu ratio in the labor market. Utilizing the fact that the optimal firm efficiently utilizes existent employment when firing cost does not exist, we simplify our representation by assuming the sum of the employment in the production and hiring sectors equates the total employment,[5] i.e. $\hat{l} + \tilde{l} = l$ where l is the total employment of the firm. g satisfies similar assumptions as f, i.e. $g' > 0$, $g'' < 0$, $g(0;\theta) = 0$ and also Inada conditions. The decreasing returns of $g(\cdot)$ induces convex hiring cost. The employees separate from the firm at instantaneous natural seperation rate $\sigma_t > 0$ at time t. In the following, time of variables is generally denoted by a subscript, however it may be expressed as an explicit argument of a function where notation becomes cumbersome. When further reduction of labor is preferable, the firm retains an option to fire workers with no cost. The firing rate is denoted by $x_t \geq 0$. The transition of labor becomes

$$\dot{l}_t = g(\tilde{l}_t;\theta_t) - \sigma_t \, l_t - x_t. \tag{1}$$

The real wage rate is denoted by a right-continuous function of time $w_t > 0$. Here, wage bargaining is assumed to be made at the industry level as is true in some economies to get a simpler view of the model. Namely, the wage rate is exogenous for the firm. Reflecting the characteristics of bargaining, the wage outcome is assumed to be less than the marginal productivity of labor in equilibrium, i.e. $w_t \leq f'(\hat{l}_t)$. Bargaining *within* the firm makes w_t a function of the state variable l, as the value of the coalition is divided between the firm and workers, which affects not only the distribution between them but also the production level when the demand constraint is not binding. This additional source of inefficiency is eliminated here to clarify the discontinuous behavior of the firm on the margin of firing phases. The firm discounts future real profits by instantaneous rate $r > 0$. Denote the value of the representative firm at time t with employment l by $\Pi(t, l)$. Normalizing output price to one and taking the initial time to zero, it becomes

$$\Pi(0, l_0) = \max_{\hat{l}, x} \int_0^\infty \Big(f(\hat{l}_t) - w_t \, l_t - c \Big) \, \mathrm{e}^{-rt} \mathrm{d}t \tag{2}$$

[5]This assumption will be removed as we introduce the firing cost later.

which is subject to the labor transition (1) and a potentially binding demand constraint

$$f(\hat{l}_t) \leq y_t \tag{3}$$

where $y_t > 0$ is the demand directed to the firm which reflects the aggregate demand in the equilibrium path. The condition in which constraint (3) becomes effective will be argued in Section II.C. Control variables are subject to the following constraints:

$$0 \leq \hat{l}_t \leq l_t \tag{4}$$

$$0 \leq x_t \tag{5}$$

Two points need to be mentioned. First, the demand constraint (3) appears since its complement is excluded as a suboptimal domain in a larger problem where off-equilibrium manuplation of output price is allowed. The assumption of differentiated outputs is only effective in this context to bring the emergence of the constraint. In a word, when all firms cannot immediately adjust outputs, the off-equilibrium attempt turns out to be unprofitable under fairly natural assumptions for a non-atomic firm to deprive of output demand from other firms via reduction of nominal output price, even though the equilibrium output growth is lower than the socially efficient level. It requires coordinated move by firms of positive measure to increase output breaking the binding constraint. It will be described below more precisely after establishing some optimality conditions. Reflecting the optimal production of firms, y is assumed to be right-continuous and right-differentiable in terms of time. Note that y is an equilibrium path which can be interpreted as effective demand in the sense that the demand is not just planned but endorsed by purchasing power in contrast to off-equilibrium paths. Second, firing consraint (5) cannot have meaningful finite upper bound from an economic view point. On the other hand, the linearity of firing x in the transition equation indicates the possibility of bang-bang control. Namely, optimal firing can diverge to infinity when $x > 0$ happens. It will turn out that such massive firing can only occur at the initial time and the time when binding y shows discoutinuous downfall, both of which are reaction to nonautonomous move. Autonomous firing is always singular.[6] Those at most countable discontinuities are handled by partitioning the problem into subperiods. Thus, firing constraint (5) is left with no upper bounds.

The above properties of firing characterizes the dynamics of equilibrim employment and production. At any time where either there is no firing or firing is singular, the output is time-differentiable and so is equilibrium y. However, we need to incorporate the costate discontinuity at the mergin of firing phases which singularity of autonomous firing brings about, and also discontinuous downfall of

[6]For a singular control problem, see Johnson and Gibson (1963) and Kelly, Kopp and Moyer (1967).

equilibrium output itself. Therefore, y is defined in the class of right differentiable functions.

The proof of the following claim will be represented after optimal conditions are characterized in Section V.

CLAIM 1: *Suppose y is continuously differentiable except for the junction times from/to the firing phase and for the time y shows discontinuity. Suppose x faces a constraint $x \in [0, \bar{x}]$ where $\bar{x} < \infty$. If $\bar{x}$ is set sufficiently large corresponding to the maximum variation of y, the optimal x is never bound by $\bar{x}$ except for the initial time and the time where y shows downward discontinuity.*

Costate variable of l is denoted by λ, Lagrangean of demand constraint by $\mu \geq 0$ and that of $\hat{l} \leq l$ by $\eta \geq 0$. The optimal dynamics of the costate variable becomes

$$\dot{\lambda} = -\left[g'(l - \hat{l}) - (r + \sigma)\right]\lambda + w - \eta. \tag{6}$$

for any time at which l is differentiable. Since y is assumed to be strictly positive and f satisfies Inada conditions, $\hat{l} > 0$ always holds. Then, the first order condition becomes

$$(1 - \mu)\, f'(\hat{l}) = \lambda\, g'(l - \hat{l}) + \eta \tag{7}$$

and for firing, by imposing arbitrarily large $\bar{x} > 0$,

$$x = \begin{cases} 0 & \text{if } \lambda > 0 \\ [0, \bar{x}] & \text{if } \lambda = 0 \\ \bar{x} & \text{if } \lambda < 0. \end{cases} \tag{8}$$

Together with (6) and (7), the optimal condition when $\hat{l} \leq l$ is not binding can be expressed as

$$(r + \sigma)\, \lambda = (1 - \mu)\, f'(\hat{l}) - w + \dot{\lambda}. \tag{9}$$

Viewing the value of the firm as an asset, the above equation says its marginal cost equates the sum of instantaneous effective marginal profits and marginal capital gain. Or, integrating (9) through l obtains $(r + \sigma)\,\Pi = \int (1 - \mu)\, f'(\hat{l})\, \mathrm{d}l - wl + \dot{\Pi}$, the left-hand side of which is the corporate cost of ownership with separation premium σ and the right-hand side is the instantaneous profits plus capital gain where the revenue is discounted by the shadow price of the demand constraint.

The dynamics of shadow price μ of the demand constraint, the impact of unexpected variation of y on the value of the firm, and marginal value of labor λ play significant roles especially on the mergin of firing phases. The following proposition guarantees that they are time-continuous in the non-firing phases.

PROPOSITION 1: *If $\lambda > 0$, then μ is time-continuous and λ is continuously differentiable on the optimal path.*

PROOF:

If demand constraint is not binding in any open domain in time, then $\mu \equiv 0$ on that domain so time-continuity of μ on the domain is obvious. Suppose demand constraint is binding in any open domain in time. Perturbation specification of the problem à la Bryson, Denham and Dreyfus (1963) would be

$$\begin{aligned}\mathrm{d}\Pi(t_i, l_{t_i}) = &\int_{t_i}^{t_{i+1}} \left(\frac{\partial H}{\partial l}\delta l + \frac{\partial H}{\partial \hat{l}}\delta \hat{l} + \frac{\partial H}{\partial x}\delta x\right) \mathrm{e}^{-r(t-t_i)}\mathrm{d}t \\ &- \int_{t_i}^{t_{i+1}} \lambda\, \mathrm{e}^{-r(t-t_i)}\, \delta \dot{l}\, \mathrm{d}t + \left(\frac{\partial \Pi(T, l_{t_{i+1}})}{\partial l_{t_{i+1}}} - \lambda_{t_{i+1}}\right) \mathrm{e}^{-r(t_{i+1}-t_i)}\delta l_{t_{i+1}}\end{aligned}$$

which should be stationary at optimum for i-th subinterval $(i = 0, 1, 2, \ldots, N)$ after dividing $[0, \infty)$ into N finite subintervals where H is the current value Hamiltonian, $t_0 = 0$ and $t_{N+1} = \infty$. To minimize the subdivision of time, let N be the number of indifferentiability of l in the time scale so that l is indifferentiable on t_i. This subdivision of time scale is necessary because the second term of the right hand side of the above equation needs to be integrable by parts to obtain (6). It requires the time derivative of l to exist. Namely, (6) is applicable only to the interior time of each subinterval. Now, choose $t \in \{s : \lambda_s > 0\}$. Then, $x = 0$ and $\hat{l}$ is differentiable from $\hat{l} = f^{-1}(y)$ and differentiability of y, which implies l is differentiable from (1). It implies t is interior time of subintervals and indeterminate function λ_t can be safely defined as a differentiable function via integration by parts of the second term of the above equation so that (6) holds. Then, the first-order condition (7) implies time-differentiability of μ on the domain. Finally, suppose that the demand constraint is binding at time T under consideration and there exists a point where the constraint is unbinding in any open neighborhood of T with radius $\varepsilon > 0$, which is denoted by $N_\varepsilon(T)$. Then, corresponding to a sequence of ε converging to zero, we can take converging subsequence of time $t \in N_\varepsilon(T)$ at any of which $f'(\hat{l}_t) = \lambda\, g'(l_t - \hat{l}_t)$ holds from the first-order condition (7) when $\lambda > 0$ and also $f(\hat{l}_t) \leq y_t$ holds. On the other hand, if $\mu_T > 0$, then (7) implies $\hat{l}_T < \lim_{t\to T} \hat{l}_t$ from the continuity of l and λ. Since $y_T = f(\hat{l}_T) < \lim_{t\to T} f(\hat{l}_t) \leq \lim_{t\to T} y_t$, it contradicts to the continuity of y. Therefore $\mu_T = 0$. It implies that μ is continuously connected at time T when transiting from domain $\mu > 0$ to $\mu = 0$ or vice versa. Finally, (6) and (7) obtain $\dot{\lambda} = (r + \sigma)\lambda - [(1 - \mu)f'(\hat{l}) - w]$. If μ is continuous, λ is continuously differentiable. □

C. Emergence condition of the demand constraint

To see the condition that breaking the demand constraint (3) becomes generally unprofitable and thus it is excluded from the admissible controls, extend the problem to allow the firm to set its individual nominal output price and focus on the time immediately after entering the constraint. Namely, the optimand (2) becomes

$$\Pi(0, l_0) = \max_{p,\hat{l},x} \int_0^\infty \left(pf(\hat{l}_t) - w_t\, l_t - c \right) \mathrm{e}^{-rt} \mathrm{d}t \tag{2'}$$

with the same constraints (4) and (5) as the original reduced problem except that (3) does not exist. In the same line of argument as Kaneko (1982) and Masso and Rosenthal (1989), the action by a single firm is indiscernible by other firms so that retaliation against deviation from a strategy is impossible.

The change of output price by the non-atomic firm producing good i alone does not affect the budget constraint of a representative consumer. Its impact on i-th firm revenue is

$$\frac{\mathrm{d}pq}{\mathrm{d}p} = q\left(1 - \frac{1}{R_r}\right)$$

where p is the output price the firm sets, q is the quantity of outputs, R_r is the coefficient of relative risk aversion. Also, the impact on the cost is

$$\frac{\mathrm{d}wl}{\mathrm{d}p} \le \frac{\mathrm{d}w\hat{l}}{\mathrm{d}p} = w\frac{\mathrm{d}q/\mathrm{d}p}{\mathrm{d}q/\mathrm{d}\hat{l}} = -\frac{wq}{pf'(\hat{l})}\frac{1}{R_r}$$

where wage rate w is real against market output prices. Then, the effect of the price change on the instantaneous profits becomes

$$\frac{\mathrm{d}\pi}{\mathrm{d}p} \ge q\left[1 - \left(1 - \frac{w}{f'\big(f^{-1}(q)\big)}\right)\frac{1}{R_r}\right] \tag{10}$$

when $p = 1$. Defining the merginal share of profits by $\alpha := 1 - w/f'(\hat{l}) \ge 1 - w/f'(l) \ge 0$, the condition

$$R_r \ge \alpha, \tag{11}$$

becomes the sufficient condition for $\mathrm{d}\pi/\mathrm{d}p \ge 0$ to hold as far as $u'' < 0$. Obviously, (11) holds in all domain if $w = f'(\hat{l})$. Also, note that $R_r = 0$ when there is no differentiation between output goods and R_r increases as outputs are differentiated so that the elasticity of substitution decreases.

The increased production capacity needs to be absorbed by increased demand brought by ceaseless discount of the output price. From (10), the lower bound of the profit change increases with q. Given condition (11) holds, the best deviation is to infinitesimaly discount the price and to produce corresponding amount at least for some time. Suppose the stopping time of the deviation is T and the firm fires all additional employment used for the deviation at time T. Integrating the optimal condition (9) in off-equilibrium context gives

$$\lambda = \int_0^T \left(pf'(\hat{l}) - w\right) \mathrm{e}^{-(r+\sigma)t}\mathrm{d}t = -\int_0^T R_r \frac{pf'(\hat{l})}{f(\hat{l})} \frac{\mathrm{d}\pi}{\mathrm{d}p} \mathrm{e}^{-(r+\sigma)t}\mathrm{d}t.$$

The right hand side goes negative if (11) holds. Therefore, (11) is the sufficient condition for profitable deviation from equilibrium becomes impossible.

It is worthwhile examining what happens in the case of non-differentiated output goods, i.e. the case of $R_r = 0$. The existnce of search friction and the limitation of hiring technology indicate $w\mathrm{d}l/\mathrm{d}p < \infty$ which implies $\mathrm{d}\pi/\mathrm{d}p = -\infty$. In such a case, reduced output price will attract infinite amount of demand which grants the attempt of constraint violation. All firms will do so, which implies that contemporaneous real wages settled by bargaining are unchanged, demand constraint is broken, and ex-post interest rate rises only by ignorable degree. The above argument suggests that the existence of demand constraint is peculiar to differentiable goods such as industrialized ones. Homogeneous goods, represented by agricultural goods for example, will not confront the constraint.

III. Steady states

This section is for completion. When the state of coordination determines the equilibrium path, steady states lose foundation for being a main focus. An eternally unbinding path converges to a saddle-point steady state. Even though the convex hiring cost prevents a jump to the steady state, it keeps a status as a good approximation in the long run. However, this is not the case for binding paths. They need specify how they are bound. There can be multiple paths — very different ones— corresponding to the same binding steady state. One may be binding forever after some point of time, one may become binding periodically. The degree of binding in steady states is no more than statistics of how the constraint binds in "transition". By this reason, it is more useful that the degree of coordination should be represented by a function of time, rather than by a value in the terminal condition. $y(t)$ defined in the previous sector is thus chosen to represent the state of coordination over time and defined to be the on-path output level in a given equilibrium. Whether the constraint is binding or not is only distinguished by its shadow price in equilibrium since the output level always coincides $y(t)$. Note that the introduction of a coordination factor transforms the system into a non-autonomous one.

Nevertheless, steady states are worth studying since, with the additional assumption of stationary belief, they give simple ordering of equilibria within the class and enables comparative statics in the long run. Both efficient and inefficient steady states are studied below. The demand constraint of the former is unbinding whereas that of the latter is binding. Steady states occur with $\lambda > 0$ and with no firing. They are ranked by the level of the demand constraint y, a representation of the degree of coordination among firms. It is shown that in steady states with demand constraint in effect, relaxation of the constraint implies the increase of the effective marginal value of employment.[7]

1. The efficient steady state

The efficient steady state has no binding demand constraint and thus characterized by $\mu = 0$. $\dot{\lambda} = 0$ in the costate dynamics (6) gives

$$\lambda_{\text{ss}} = \frac{w}{g'(l_{\text{ss}} - \hat{l}_{\text{ss}}) - (r + \sigma)} > 0,$$

which implies $x_{\text{ss}} = 0$ where subscript "ss" represents the value at the efficient steady state. Inada conditions guarantee that the first-order condition (7) provides an interior solution when $\lambda > 0$ so we have $f'(\hat{l}_{\text{ss}}) = \lambda_{\text{ss}}\, g'(l_{\text{ss}} - \hat{l}_{\text{ss}})$. Together with $\dot{l} = 0$ in labor transition (1), $(l_{\text{ss}}, \hat{l}_{\text{ss}})$ is characterized to solve

$$f'(\hat{l}_{\text{ss}}) = \frac{g'(l_{\text{ss}} - \hat{l}_{\text{ss}})}{g'(l_{\text{ss}} - \hat{l}_{\text{ss}}) - (r + \sigma)} w \tag{12}$$

$$g(l_{\text{ss}} - \hat{l}_{\text{ss}}) = \sigma\, l_{\text{ss}}, \tag{13}$$

the solution of which exists when $r > 0$.[8] Note that (12) implies $f'(\hat{l}_{\text{ss}}) > w$. The marginal productivity of labor in the production sector is strictly greater than the marginal cost of labor simply because some labor is absorbed in the hiring activity. If hiring efficiency rises, then the difference between f' and w shrinks.

2. Inefficient steady states

Inefficient steady states are constrained steady states characterized by binding demand constraint with $\mu > 0$. They satisfy $f(\hat{l}_{\text{iss}}) = y$ where subscript "iss" represents the value at inefficient steady states. For their existence, y must be a constant function of time. With stationarity in labor transition, the inefficient steady state for given y is characterized by

$$\lambda_{\text{iss}} = \frac{w}{g'\big(l_{\text{iss}} - f^{-1}(y)\big) - (r + \sigma)} > 0 \tag{14}$$

$$g\big(l_{\text{iss}} - f^{-1}(y)\big) = \sigma\, l_{\text{iss}}. \tag{15}$$

[7] As a benchmark, in spot production with demand constraint, the effective marginal value becomes the present value of wages. It is derived from the fact that, in each moment, $\max_l f(l) - w\, l$ subject to $f(l) \le y$ implies $w = (1-\mu)\, f'(l)$ with $\mu \ge 0$. In this case, an increase of the effective marginal value of employment leads to the rise of wages.

[8] If $r \le g'(L) - \sigma < 0$ where L is the solution in $g(L) = \sigma L$ where all employment is dedicated to hiring, the solution may not exist. However, since negative discount rate is unlikely to hold in steady states, we can safely eliminate such a case. It implies $g'(L) > r + \sigma$. By decreasing returns of g, $g'(l - \hat{l}) > r + \sigma$ holds for any unconstrained/constrained steady states.

The relation between inefficient and the efficient steady states is depicted in Figure 1. The curve labeled $\dot{\lambda} = 0$ shows the costate steady state condition for an

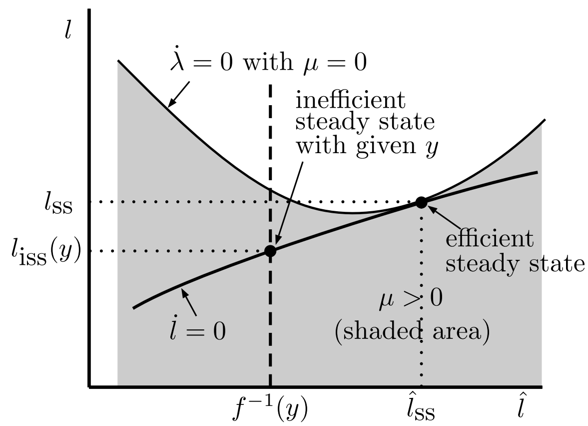


Figure 1. : The efficient and inefficient steady states

inefficient case (12). The curve labeled $\dot{l} = 0$ shows the labor steady state conditions (13) and (15). Since $f^{-1}(y) \leq \hat{l}_{\text{iss}}$, stationarity conditions for labor (13) and (15) imply that $l_{\text{iss}} \leq l_{\text{ss}}$ the inequality of which is strict except for the obvious case $f(\hat{l}_{\text{iss}}) = y$. Therefore, the curve $\dot{l} = 0$ always locates below the curve $\dot{\lambda} = 0$ for any $\hat{l} \leq \hat{l}_{\text{ss}}$. Efficient steady states locate on the curve $\dot{l} = 0$.

λ_{iss} given by (14) is expressed in another form $\lambda_{\text{iss}} = [(1-\mu)f'(\hat{l}_{\text{iss}}) - w]/(r+\sigma)$. It reads the marginal value of employment for the firm is the discounted present value of $(1-\mu)f'(\hat{l}_{\text{css}}) - w$ in which the discount rate is affected by separation premium. In this place, $(1-\mu)f'(\hat{l})$ is termed the *effective marginal productivity* of the production sector. Facing the demand constraint, effective price of output is discounted by the shadow price of the constraint. Hiring sector has costate variable λ as its internal output price, which is also indirectly affected by present and future shadow price of the demand constraint.

PROPOSITION 2: *At an inefficient steady state, increase in demand y brings 1) increase in total employment and in both production and hiring sectors, 2) increase in marginal value of employment, 3) increase in effective marginal productivity of the production sector and increase in marginal productivity of the hiring sector, and 4) decrease in demand duals.*

PROOF:

1) Applying the implicit function theorem to (15) obtains $\mathrm{d}l_{\text{iss}}/\mathrm{d}y = g'/[f'(g' - \sigma)] > 0$. On the other hand, $\mathrm{d}\hat{l}_{\text{iss}}/\mathrm{d}y = 1/f' > 0$. They result in

$$\text{(16)} \qquad \frac{\mathrm{d}\tilde{l}_{\text{iss}}}{\mathrm{d}y} = \frac{\mathrm{d}l_{\text{iss}}}{\mathrm{d}y} - \frac{\mathrm{d}\hat{l}_{\text{iss}}}{\mathrm{d}y} = \frac{\sigma}{g' - \sigma}\frac{1}{f'} > 0.$$

2) Similar application of the theorem to (14) derives $\mathrm{d}\lambda_{\text{iss}}/\mathrm{d}\tilde{l}_{\text{iss}} = -g''\lambda_{\text{iss}}^2/w$, from which (16) implies

$$\frac{\mathrm{d}\lambda_{\text{iss}}}{\mathrm{d}y} = -\frac{g''\lambda_{\text{iss}}^2}{w}\frac{\sigma}{g'-\sigma}\frac{1}{f'} > 0.$$

3) From the above result,

$$\begin{aligned}\frac{\mathrm{d}\lambda_{\text{iss}}g'(\tilde{l}_{\text{iss}})}{\mathrm{d}y} &= \left(\lambda_{\text{iss}}g'' + \frac{\mathrm{d}\lambda_{\text{iss}}}{\mathrm{d}\tilde{l}_{\text{iss}}}g'\right)\frac{\mathrm{d}\tilde{l}_{\text{iss}}}{\mathrm{d}y} \\ &= \left(1 - \frac{g'\lambda_{\text{iss}}}{w}\right)g''\lambda\frac{\mathrm{d}\tilde{l}_{\text{iss}}}{\mathrm{d}y} \\ &= (r+\sigma)\frac{\mathrm{d}\lambda_{\text{iss}}}{\mathrm{d}y} > 0\end{aligned}$$

where $\lambda_{\text{iss}}g'(\tilde{l}_{\text{iss}})$ is the marginal productivity of the hiring sector and from the first-order condition (7), it equates to the effective marginal productivity $(1-\mu)f'(\hat{l}_{\text{iss}})$.

4) Since $1-\mu_{\text{iss}} = \lambda_{\text{iss}}g'(\tilde{l}_{\text{iss}})/f'(\hat{l}_{\text{iss}})$,

$$\frac{\mathrm{d}(1-\mu_{\text{iss}})}{\mathrm{d}y} = \frac{r+\sigma}{f'}\frac{\mathrm{d}\lambda_{\text{iss}}}{\mathrm{d}y} - \frac{\lambda_{\text{iss}}g'f''}{(f')^3} > 0,$$

which brings $\mathrm{d}\mu_{\text{iss}}/\mathrm{d}y < 0$. □

The result that the effective marginal productivity and the marginal value of labor increase may worth attention. Depending on the wage bargaining process, they can imply the rise of wages. It means that increase in both wages and employment coexist without technological progress. The last result of the proposition shows that the impact of the unexpected variation of demand on the firm's value becomes smaller as the demand level becomes higher.

IV. Effect of binding demand constraint

Let $z_t = (l_t^*, \lambda_t^*)$ be the optimal path. The next proposition says, if there are binding and unbinding optimal paths which share common history after some point of time, the binding path has lower employment and lower marginal value of labor before that point.

PROPOSITION 3: *Consider two optimal paths $z_t^{(1)}$ and $z_t^{(2)}$, and denote corresponding adjoint variable to the demand constraint by $\mu^{(1)}$ and $\mu^{(2)}$, respectively. Suppose that there exists $t_0, t_1 \in \mathbb{R}$ such that $z_{t_1}^{(1)} = z_{t_1}^{(2)}$ and $z_t^{(1)}$ accompanies $\mu_t^{(1)} > 0$ for $\forall t \in [t_0, t_1)$ whereas $z_t^{(2)}$ holds with $\mu_t^{(2)} = 0$ for $\forall t \in [t_0, t_1)$. Then, there exists $t_c < t_1$ such that $z_t^{(1)} < z_t^{(2)}$ holds for any $t \in [t_c,\, t_1)$.*

PROOF:

Pick up time $(T-\varepsilon)$ for an arbitrarily small $\varepsilon > 0$. Suppose $\lambda^{(1)}_{T-\varepsilon} \geq \lambda^{(2)}_{T-\varepsilon}$. From the first order condition (7),

$$\frac{g'(\tilde{l}^{(1)}_{T-\varepsilon})}{f'(\hat{l}^{(1)}_{T-\varepsilon})} = \frac{1-\mu^{(1)}_{T-\varepsilon}}{\lambda^{(1)}_{T-\varepsilon}} < \frac{1}{\lambda^{(2)}_{T-\varepsilon}} = \frac{g'(\tilde{l}^{(2)}_{T-\varepsilon})}{f'(\hat{l}^{(2)}_{T-\varepsilon})}$$

which implies

$$\frac{\tilde{l}^{(1)}_{T-\varepsilon}}{\hat{l}^{(1)}_{T-\varepsilon}} > \frac{\tilde{l}^{(2)}_{T-\varepsilon}}{\hat{l}^{(2)}_{T-\varepsilon}} \tag{17}$$

by the decreasing returns of f and g. Then, $\tilde{l}^{(1)}_{T-\varepsilon} > \tilde{l}^{(2)}_{T-\varepsilon}$ holds, since otherwise $\dot{l}^{(1)}_{T-\varepsilon} \leq \dot{l}^{(2)}_{T-\varepsilon}$ holds which means $l^{(1)}_{T-\varepsilon} \geq l^{(2)}_{T-\varepsilon}$. Together with (17), it implies $\hat{l}^{(1)}_{T-\varepsilon} > \hat{l}^{(2)}_{T-\varepsilon}$, a contradiction. So, $\dot{l}^{(1)}_{T-\varepsilon} > \dot{l}^{(2)}_{T-\varepsilon}$ and $l^{(1)}_{T-\varepsilon} < l^{(2)}_{T-\varepsilon}$ hold. On the other hand, costate transition (6) implies $\dot{\lambda}^{(1)}_{T-\varepsilon} > \dot{\lambda}^{(2)}_{T-\varepsilon}$, i.e. $\lambda^{(1)}_{T-\varepsilon} < \lambda^{(2)}_{T-\varepsilon}$, which is a contradiction to the first assumption.

Therefore, assume $\lambda^{(1)}_{T-\varepsilon} < \lambda^{(2)}_{T-\varepsilon}$ which implies $\dot{\lambda}^{(1)}_{T-\varepsilon} > \dot{\lambda}^{(2)}_{T-\varepsilon}$ from $\lambda^{(1)}_{T} = \lambda^{(2)}_{T}$. It implies $g'(\tilde{l}) > r + \sigma$. Suppose $l^{(1)}_{T-\varepsilon} \geq l^{(2)}_{T-\varepsilon}$. Since only $z^{(1)}_{T-\varepsilon}$ is binding, $\hat{l}^{(1)}_{T-\varepsilon} < \hat{l}^{(2)}_{T-\varepsilon}$ which means $\tilde{l}^{(1)}_{T-\varepsilon} > \tilde{l}^{(2)}_{T-\varepsilon}$. Since $g'(\tilde{l}) > r + \sigma > \sigma$, it implies $\dot{l}^{(1)}_{T-\varepsilon} > \dot{l}^{(2)}_{T-\varepsilon}$, i.e. $l^{(1)}_{T-\varepsilon} < l^{(2)}_{T-\varepsilon}$, a contradiction. Therefore, $l^{(1)}_{T-\varepsilon} < l^{(2)}_{T-\varepsilon}$. By continuity of l and λ, the statement of the proposition follows. □

Note that if $z^{(1)}_{t_0} < z^{(2)}_{t_0}$ and $\mu^{(1)}_t = \mu^{(2)}_t = 0$ for $t < t_0$, then $t_c < t_0$ by continuity of l and λ. Namely, if it is expected that the demand constraint becomes binding in future, the firm refrains from hiring workers since their marginal value is low and thus retains smaller number of employment.

V. Firing

Although section IV suggests that binding demand constraint today and in future tends to bring a sluggish economic state, Proposition 1 guarantees that the presence of the demand constraint does not drastically change the structure of the model *as far as the marginal value of labor is strictly positive.* However, this is not true if it becomes zero, i.e. when workers are undergoing firing. Since the transition of labor is linear in firing, precise treatment is necessary when the switching function is on the boundary. Employment becomes redundant if both current and future production requires smaller labor reflected in y and λ. It is shown that, as far as that y is differentiable and the economy already has some history, the adjustment of labor is fulfilled within continuous variation. Namely, with feasible initial employment, firing is done with $\lambda = 0$ which implies singularity of the problem, i.e. a non-standard procedure is required to obtain the optimal path. On the other hand, the continuity of employment brings about discontinuity in the costate variable. This can be observed by considering the firing phase condition $y = f(l)$ as a terminal surface in a subproblem where the terminal condition is to decide when and where to enter in the terminal surface. The optimal terminal condition does not guarantee smooth entering.

In the following description, the following notations are used for subsets on the time scale. Define the firing phase X_t the maximum connected set on time scale including t where $x^*_s > 0$ for any $s \in X_t$ and the asterisk shows the optimal path. Formally, $X_t := \bigcup_i \{I_i \ni t : x^*_s > 0, \forall s \in I_i\}$ where I_i is a closed interval. If $x^*_t = 0$, then $X_t = \emptyset$. The closure of X_t writes in an interval form $\bar{X}_t = [t^E, t^L]$ where t^E is the entering time in X_t and t^L is the leaving time from X_t. Although they are defined interval-wise, they are used without mentioning it if the objective interval is obvious. Similarly, the non-hiring phase Λ^o_t is defined to be the maximum connected set on time scale including t where $\lambda^*_s = 0$ for any $s \in \Lambda^o_t$, i.e. $\Lambda^o_t := \bigcup_i \{I_i \ni t : \lambda^*_s = 0, \forall s \in I_i\}$. If $\lambda^*_t > 0$, then $\Lambda^o_t = \emptyset$. In the interval form, $\bar{\Lambda}^o_t = [t^e, t^l]$ where the entering and leaving time from/to Λ^o_t are denoted by t^e and t^l, respectively. Again, the same abuse of notation applies as $\bar{X}_t$. Also, define $\Lambda^-_t := \bigcup_i \{I_i \ni t : \lambda^*_s < 0, \forall s \in I_i\}$ and $W := \left\{t : l^*_t < f'^{-1}(w_t)\right\}$. Union of those intervals are denoted without subscripts by $X := \bigcup_t X_t$, $\Lambda^o := \bigcup_t \Lambda^o_t$ and $\Lambda^- := \bigcup_t \Lambda^-_t$. Denote the set of all t^E's, t^L's, t^e's and t^l's by E^X, L^X, E^Λ and L^Λ, respectively.

PROPOSITION 4: *$X_t \subset \Lambda^o_t$ for any $t \in W$.*

PROOF:

The result is obvious if $X_t = \emptyset$. Suppose $X_t \neq \emptyset$. Define $\bar{x} = \sup_t(-\dot{y}_t/f'(l_t))$. Suppose that the demand constraint holds with equality, i.e. $f(\hat{l}_t) = y_t$ for some $t \in \Lambda^-_t \cap W$ such that $\Lambda^-_t \neq \emptyset$. Optimal $x_t = \bar{x}$ leads to $\mu_s = 0$ for any $s \in \Lambda^-_t$ such that $s > t$ since $f'(l_t)\dot{l}_t < \dot{y}_t$. Since μ_s is right-continuous, $\mu_t = 0$ and $\dot{\lambda}_s$ exists for such s and $t + 0$.[9] From (6) and (7), costate transition becomes $\dot{\lambda}_s = (r+\sigma)\lambda_s - f'(l_s) + w_s < 0$ for any $s \geq t$. The relation is recursively justified starting from $s = t$ so that $s \in \Lambda^-_t$ implies $\dot{l}_s < 0$ which means $s + \varepsilon \in W$ and thus $s + \varepsilon \in \Lambda^-_t \cap W$ for arbitrarily small $\varepsilon \geq 0$. It ultimately causes $\lim_{s\to\infty} l_s < 0$. Therefore, any path which enters $\Lambda^-_t \cap W$ cannot be optimal. It implies that $X_t \cap W \not\subseteq \Lambda^-_t \cap W$ whereas $X_t \subseteq \Lambda^o_t \cup \Lambda^-_t$ for any $X_t \neq \emptyset$. Therefore, $X_t \cap W \subseteq \Lambda^o_t \cap W$. □

The above proposition proves Claim 1 and firing x becomes a singular control, not a bang-bang:

COROLLARY 1: *Firing x_t is a singular control if $t \in W$.*

[9]See Theorems 4.1 and 4.2 of Hartl, Sethi and Vickson (1995).

PROOF:
Since l is right-differentiable, $|X_t| > 0$ if $X_t \neq \emptyset$. From Proposition 4, it implies $|\Lambda_t^o| > 0$. Namely, if $x_t > 0$, then $\lambda_t = 0$ and $\lambda_{t+\varepsilon} = 0$ for arbitrarily small $\varepsilon > 0$. Since λ_t is a switching function of x_t, which is zero on an interval with positive measure whenever $x_t > 0$, x_t is a singular control. □

The proposition suggests that, for a massive firing, i.e. discontinuous decrease of employment, to happen, it should occur only once at the very beginning of the economy, which implies that we can safely separate such a phase from the analysis and concentrate on the dynamics after the negative "big-bang". Since $\bar{x}$ is an artificial boundary, if such a transition were to happen, it finishes instantaneously by a discontinuous decrease of employment, which can be understood as the limit of dynamics when $\bar{x} \to \infty$. Even if continuously large non-autonomous force acts in the middle, it will not bring the system to the initial big-bang state since sufficiently large continuous change of employment absorbs such a shock. Figure 2 shows a phase diagram for an unbounded autonomous case. The manifold

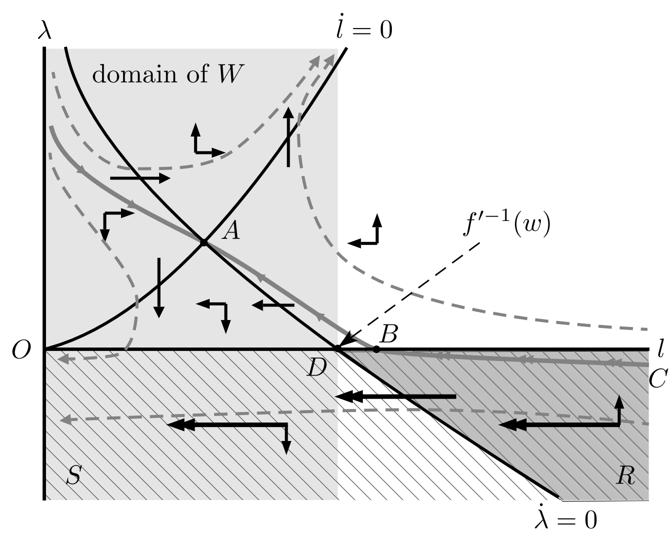


Figure 2. : Phase diagram for an unbounded autonomous case

$\dot{l} = 0$ is an upward sloping curve stable in terms of l that passes through the neighborhood of the origin. That of $\dot{\lambda} = 0$ is downward sloping, unstable in terms of λ, so that $\lambda \to +\infty$ holds as $l \to 0$ on it and passes through $l = f'^{-1}(w)$. In the forth quadrant shown as a hatched area, there is "strong" leftward flows. The flow instantaneously reaches to the goal as $\bar{x} \to \infty$. There is a saddle path to the unique non-zero steady state A in this particular unbounded autonomous case. The saddle path in the forth quadrant is drawn as almost flat downward sloping curve in the figure. However, since $\bar{x}$ is arbitrary, it should be understood that it converges to the horizontal axis as $\bar{x} \to \infty$, the optimal firing on the path is $x^* \to +\infty$. The same holds for any paths which reaches to any points on the horizontal axis except interval OD. In non-autonomous case, the optimal path is generally different from the one drawn in the diagram because phase diagram itself transforms. If the optimal path in such a case were in the hatched light grey area S in Figure 2, the only way that it survives as optimal is to move to the hatched dark grey area R. In other cases, the path trespasses on the negative employment region due to $\dot{\lambda} < 0$. Move from S to R is generally impossible in the autonomous case as the vector field in the figure shows. However, non-autonomous cases require a check whether sufficiently quick fluctuation of the boundary of domain of W caused by external forces does not actually allow such a move. If allowed, the path may come back to the non-hatched area surviving as optimal. The following proposition formalizes that it never happens even in non-autonomous cases.

PROPOSITION 5 (Impossibility of a jump in the middle): *If $t \in W$, then $\lambda_s \geq 0$ for any $s > t$ along the optimal path.*

PROOF:
Suppose $\lambda_v = 0$ at $v > t$ and $\lambda_{v+\varepsilon} < 0$ for arbitrarily small $\varepsilon > 0$. Then, from the transition (6) of λ, right-continuity of μ and continuity of w and l, $(1-\mu_{v+\varepsilon})f'(l_{v+\varepsilon}) > w_{v+\varepsilon}$ must hold for $\lambda_{v+\varepsilon} < 0$, which implies $v+\varepsilon \in W$. Then, since x is arbitrarily large in (1), $\dot{l}$ is always smaller than any non-autonomous change of $f'^{-1}(w)$, which implies the optimal path never enters W^c from $W \cap \Lambda^-_{v+\varepsilon}$, thus diverging to $l_s \to -\infty$ as $s \to \infty$. Thus, it is excluded from the optimal path. □

The proposition can be restated as follows with the optimal control (8).

COROLLARY 2: *$x_t > 0$ occurs only if $\lambda_t = 0$ for any $t \in W$ along the optimal path.*

Based on the singularity of firing, the next proposition shows that it occurs only when the demand constraint is binding and $\dot{y} < 0$.

PROPOSITION 6 (Optimal controls and binding demand constraint): *$\hat{l}^*_t = l^*_t$ and $\tilde{l}^*_t = 0$ for $t \in \Lambda^o \cap W$.*

(18) $$x_t = -\sigma l_t - \frac{\dot{y}_t}{f'(l_t)}$$

and $\mu_t > 0$ for any $t \in (\mathring{\Lambda}^o \cup E^\Lambda) \cap W$. Moreover, μ_t is differentiable for any $t \in \mathring{\Lambda}^o \cap W$ and right-differentiable for $t \in E^\Lambda$. Also

(19) $$f(l^*_t) = y^*_t$$

for any $t \in \bar{\Lambda}^o \cap W$ and $\dot{y}_t < 0$ for any $t \in \Lambda^o \cap W$.

PROOF:
$\lambda_t = \dot{\lambda}_t = 0$ for any $t \in \Lambda^o \cap W$ where $\dot{\lambda}$ is right-derivative at $t = t^e$ and left-derivative at $t = t^l$. Then, costate dynamics (6) implies $\eta = w > 0$ and also $\hat{l} = l$ and $\tilde{l} = 0$ by complementarity. The first order condition (7) obtains

$$(1-\mu)\, f'(l) = \eta = w. \tag{20}$$

It implies $\mu_t > 0$ for any $t \in \mathring{\Lambda}^o \cap W$ since $f'(l_t) > w_t$ for $t \in W$. (20) holds for $t^e \in W$ from the right-continuity of μ and the continuity of l and w, so $\mu_{t^e} > 0$. Together with continuity of l, it implies that $f(l_t^*) = y_t$ holds for any $t \in \bar{\Lambda}^o \cap W$. Since $\dot{l}_t = -\sigma l_t - x_t = \dot{y}_t / f'(l_t)$ holds for any $t \in \Lambda^o \cap W$, optimal firing is given by (18). Since $x_t \geq 0$, it implies $\dot{y}_t \leq -\sigma l_t f'(l_t) < 0$. It also implies differentiability of μ_t for any $t \in \mathring{\Lambda}^o \cap W$ and right-differentiability for $t \in E^\Lambda$ by the differentiability of l and w and $f \in C^1$. □

Suppose that $B_t := \Lambda_t^o \setminus X_t \neq \emptyset$ for some t and $s \in B_t$. If B_t consists of an interval, then Proposition 6 implies $\dot{l} = -\sigma l$ and thus $\dot{y} = -\sigma l f'(l)$ need to hold on that interval. However, it is generically true that actual y will not satisfy this condition so that we can safely assume $\bar{X}_t = \Lambda_t^o$ almost surely if both are not empty and if y is considered to be randomly chosen by nature.

VI. Entering and leaving from the non-hiring phase

Proposition 6 brings a similar situation as optimal control problems with state variable inequality constraints (SVICs). A state constraint equivalent to (19) in SVICs would have worked as a binding constraint on controls only in its derivative form as $f'\dot{l} = \dot{y}$. By nature of the derivative form, it does not tell alone when the constraint becomes binding or off-binding, which requires additional information on the level. In the current problem, binding constraint (19) is not given but derived from optimal conditions, however, the same property holds. The jump condition is derived from the truncatability of the problem into subperiods Λ^o and $(\Lambda^o)^c$ as do those in SVICs. Whenever $E^\Lambda \neq \emptyset$, the truncated problem at the initial time $0 \notin \Lambda^o$ can be regarded as a problem with the terminal surface (19), where the terminal time t^e and the terminal state $l(t^e)$ are to be optimally determined. The problem brings the terminal costate variable to be $\lambda_{t^e} > 0$. Since $\lambda = 0$ in Λ^o, it implies costate discontinuity at entering time. The same holds for the leaving time.

Suppose $E^\Lambda \neq \emptyset$ and $0 \notin \Lambda^o$. Let $t^e \in E^\Lambda$ be the first entering time to Λ^o. Regarding t^e as the terminal time in discretion, the truncated problem as of time zero rewrites as follows.

$$\Pi(0, l_0) = \max_{\hat{l}, x, t^e} \int_0^{t^e} \left(f(\hat{l}) - w\, l - c \right) \mathrm{e}^{-rt} \mathrm{d}t + \Pi(t^e, l(t^e))\, \mathrm{e}^{-rt^e} \tag{21}$$

The state transition and constraints remain the same as before. Note that the firm retains an option to make a massive firing with no cost at the entering time. Thus, the terminal condition is given by an inequality

$$f\big(l(t^e)\big) \geq y(t^e). \tag{22}$$

Define the terminal-time Lagrangean $\Phi(t, l_t)$ by

$$\Phi(t, l_t) := \Pi(t, l_t) + \mu(t^e)\,(f(l_t) - y_t)$$

where $\mu(t^e)$ is the Lagrange multiplier attached to the terminal surface. Define general notations $z(T_-) := \lim_{t\uparrow T} z(t)$ and $z(T_+) := \lim_{t\downarrow T} z(t)$ for any time-dependent variable z. Then, the terminal condition on the costate variable becomes

$$\lambda(t^e_-) = \frac{\partial \Phi}{\partial l} = \frac{\partial \Pi}{\partial l} + \mu(t^e) f'.$$

Since $\partial\Pi/\partial l = \lambda(t^e) = 0$, it implies

$$\lambda(t^e_-) = \mu(t^e)\, f'\big(l(t^e)\big). \tag{23}$$

The condition on terminal time is given by

$$f\big(\hat{l}(t^e_-)\big) - y + \lambda(t^e_-)\left[g(\tilde{l}(t^e_-)) - \sigma l\right] = r\Pi(t^e, l(t^e)) + \mu(t^e)\,\dot{y}. \tag{24}$$

(23) and (24) settle the relation between the entering time and the costate variable as follows.

$$\left(\dot{l}_-\big(\lambda(t^e_-)\big) - \dot{l}_+\right)\lambda(t^e_-) = y - f\big(\hat{l}(\lambda(t^e_-))\big) + r\Pi(t^e, l(t^e)) \tag{25}$$

where $\dot{l}_-\big(\lambda(t^e_-)\big) = g(\tilde{l}_-(\lambda_-)) - \sigma l$ and $\dot{l}_+ = \dot{y}/f'\big(l(t^e)\big)$. Note that $\dot{l}_-$ is a function of $\lambda(t^e_-)$. This condition can be interpreted as

$$\frac{\partial \Pi}{\partial t} = r\Pi + (y - wl - c) + \dot{l}_+ \frac{\partial \Pi}{\partial l}.$$

Namely, *the direct benefit of postponing the entering time equates the return of the firm's value plus instantaneous profit plus the increased value caused by change of employment.*

The following proposition guarantees that no massive firing occurs at the entering time. Namely, (22) does not hold with inequality. Also, it implies that the demand constraint must be unbinding before entering the firing phase. It is possible for the demand constraint to be binding without accompanying firing, however, it will not start firing without leaving the demand constraint once. Note that the presence of labor search friction raises strictly positive rent in a match between a firm and workers. As far as the distribution rule between the two parties is not dominated by workers, the firm receives strictly positive share of rent, i.e. $\Pi > 0$. So the assumption in the proposition generally holds in an economy with labor search friciton.

PROPOSITION 7: *Suppose* $t^e - \varepsilon \in W$ *for any* $\varepsilon > 0$. *If* $\Pi\big(t^e_-, l(t^e)\big) > 0$, *then* $\lambda(t^e_-) > 0$, $\mu(t^e_-) = 0$, $\mu(t^e) > 0$ *and* $f\big(l(t^e)\big) = y(t^e)$.

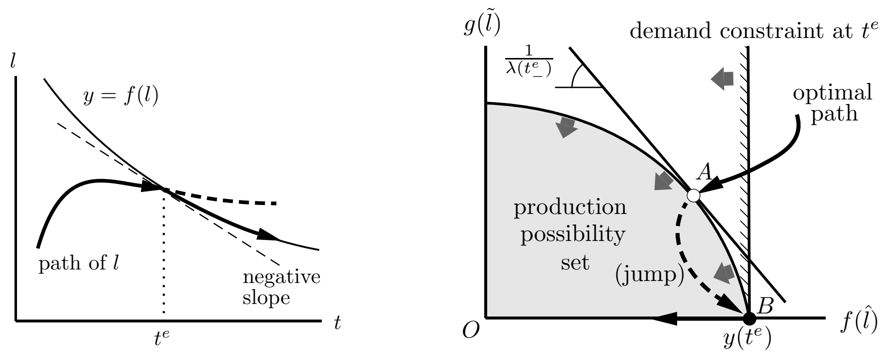
PROOF:
$\lambda(t^e_-) < 0$ is impossible from (23). Suppose $\lambda(t^e_-) = 0$. Then, $\tilde{l}(t^e_-) = 0$ and $\hat{l}(t^e_-) = l(t^e_-)$ from (7) and $\mu(t^e) = 0$ from (23). Then, (24) becomes

$$f\big(l(t^e_-)\big) - y(t^e_-) = r\Pi(t^e_-, l(t^e)) > 0. \tag{24'}$$

Then, $f\big(\hat{l}(t^e_-)\big) \leq y(t^e_-) < f\big(l(t^e_-)\big)$ implying $\tilde{l}(t^e_-) > 0$, which contradicts the optimality when $\lambda(t^e_-) = 0$. Therefore, $\lambda(t^e_-) > 0$.

Applying the above result to (23) obtains $\mu(t^e) > 0$. Also, applying to (25) implies $\dot{l}_-\big(\lambda(t^e_-)\big) > \dot{l}_+$, namely $f'\big(l(t^e)\big)\,\dot{l}_-(t^e_-) > \dot{y}$. Since $f(l(t^e)) = y(t^e)$ from Proposition 6 and $\hat{l}(t^e_-) < l(t^e_-)$ hold, it implies $f(\hat{l}(t^e_-)) < f(l(t^e_-)) = y$, thus $\mu(t^e_-) = 0$. $\square$

Since $\lambda(t^e_+) = 0$, Proposition 7 implies general discontinuity of λ at entering time for an economy with $\Pi \not\equiv 0$. The second half of the proposition means that the path of l "bumps" into the demand surface as shown in Figure 3a. The effect of the same costate discontinuity is reflected in the diagram of production possibility set in Figure 3b. The optimal path goes into point A on the production frontier where shadow price of hiring λ is strictly positive at the entering time and jumps to point B. Since B is bound by the demand constraint as Proposition 6 predicts, point A must be unbound because $\hat{l}(t^e_-) < \hat{l}(t^e_+)$. The entering behavior



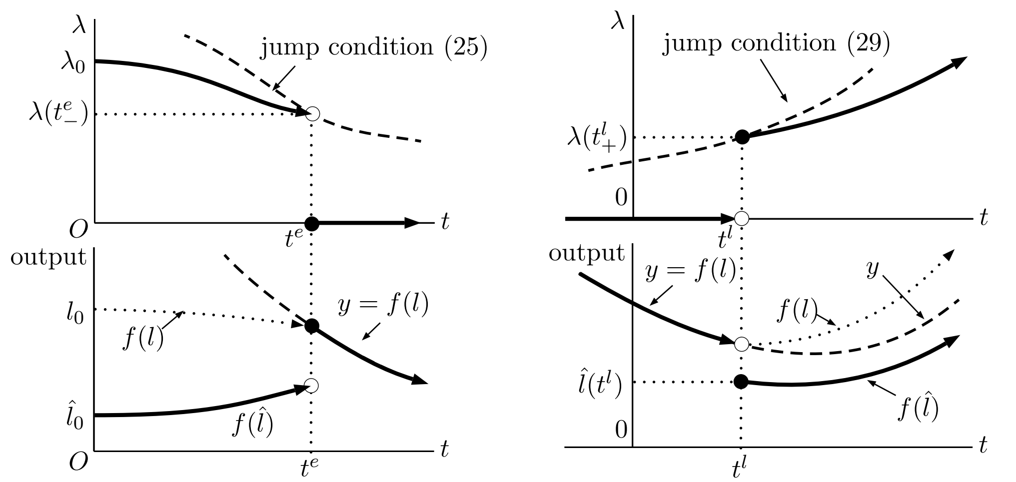
(a) Kink in the employment path (b) Jump on production frontier

Figure 3. : Entering behavior

is summarized in a literate manner as follows.

PROPOSITION 8: *At entering time, production discontinuously increases so that unbinding demand constraint beforehand becomes binding afterwards. Hiring discontinuously decreases from strictly positive to zero. Costate variable λ jumps from strictly positive to zero. Demand dual μ jumps from zero to strictly positive. Time path of l kinks so that $\dot{l}$ jumps downwards.*

Figure 4a draws how the jump condition (25) and the terminal surface (22) determine the first entering time. Starting from the initial employment l_0 and the hypothetical initial costate value λ_0, transitional equations (1) and (6) govern the dynamics. The entering time must satisfy the jump condition (25) which is drawn as a broken curve on the above plane. The time when the path of λ encounters the surface of the jump condition is the entering time. At this time, the production jumps from $f(\hat{l})$ to $f(l)$ and the latter coincides with y. It only holds for the correct initial costate value. If not, the initial hypothesis on λ must be corrected. For the rest of entering times, if exist, λ_0 and l_0 should be replaced by $\lambda(t^l_+)$ and $l(t^l)$ where t^l is the previous leaving time.



(a) Entering (b) Leaving

Figure 4. : Determining junction time

The leaving behavior is analyzed in a similar fashion. Now, consider a production decision as of t^e. Again, the problem is truncated at $t^l \in \bar{\Lambda}^o_{t^e}$ to check the differentiability at the interface of the firing phase. As we have already obtained the optimal policy on Λ^o in section V, the remaining problem is the choice of t^l, which is obtained from the terminal condition. The problem is the same as (21) except that the initial and terminal time is replaced by t^e and t^l, respectively, and the discounting period is modified to the interval starting from t^e. So, after applying the optimal policy on Λ^o, we have

$$\Pi(t^e, l(t^e)) = \max_{t^l} \int_{t^e}^{t^l} \big(y - wf^{-1}(y) - c\big)\, \mathrm{e}^{-r(t-t^e)}\mathrm{d}t + \Pi(t^l, l_{t^l})\, \mathrm{e}^{-r(t^l-t^e)}.$$

Transitions and constraints remain the same. At the leaving time, the following terminal constraint must hold.

$$f(l_{t^l}) = y_{t^l} \tag{26}$$

This is required before optimization is undertaken, since Proposition 6 tells that the effectiveness of the constraint (19) is derived from optimality only for $t \in \mathring{\Lambda}^o \cap W$ but it extends to $t \in \bar{\Lambda}^o \cap W$ by *continuity* of l. However, one may find subtlety in inclusion of this terminal constraint in the problem. So let us check its validity. First, we proceed with the constraint. The terminal condition becomes

$$-\Big[f\big(\hat{l}(t^l)\big) - f\big(l(t^l_-)\big)\Big] - \lambda(t^l)\,\Big[g(\tilde{l}(t^l) - \sigma l\Big] = r\Pi\big(t^l, f^{-1}(y_{t^l})\big) - \nu^l \dot{y} \tag{27}$$

$$\lambda(t^l) = \nu^l f'\big(l(t^l)\big) \tag{28}$$

where ν^l is the Lagrange multiplier adjoint to the terminal constraint. If there were not a terminal condition, we can set $\nu^l = 0$ which implies that $\lambda(t^l)$ becomes continuous at zero. By imposing $\nu^l = \lambda(t^l) = 0$, we have $f\big(l(t^l_-)\big) - f\big(\hat{l}(t^l)\big) = r\Pi\big(t^l, f^{-1}(y_{t^l})\big)$. Since $\lambda_{t^l} = 0$, we have $\hat{l}(t^l) = l(t^l)$ implying $\Pi\big(t^l, f^{-1}(y_{t^l})\big) = 0$. This is impossible to happen as argued in the proof of Proposition 7. Therefore, the terminal constraint (26) is required for the economy to exist within the framework of the maximum principle. Since $\nu = 0$ is impossible in (28), it also showed the following.

PROPOSITION 9: *λ is discontinuous at the leaving time so that $\lambda(t^l_-) = 0$ and $\lambda(t^l) > 0$ hold.*

Next, derive the jump condition for leaving. From (27) and (28), we obtain

$$\Big(\dot{l}_+(\lambda_+) - \dot{l}_-\Big)\,\lambda_+ = y - f(\hat{l}_+)) - r\Pi(t^l, l(t^l)). \tag{29}$$

where $\dot{l}_+(\lambda_+) = g\big(\tilde{l}_+(\lambda_+)\big) - \sigma l$ and $\dot{l}_- = \dot{y}/f'\big(l(t^l)\big)$. Note that $\dot{l}(t^l_+)$ is a function of $\lambda(t^l_+)$. We can rewrite (29) as

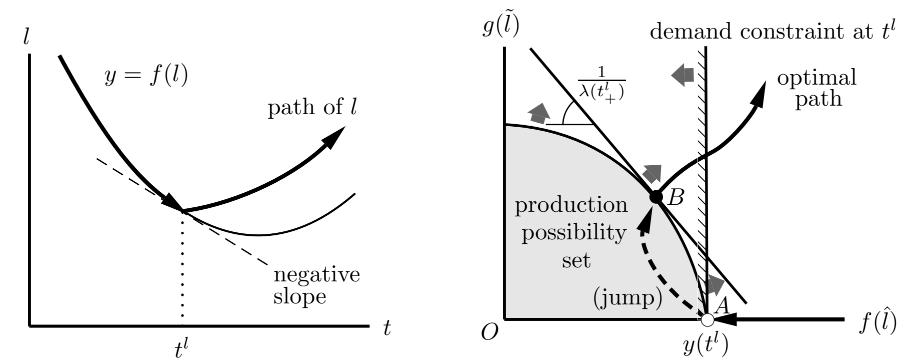
$$y - wl + \dot{l}(t^l_-)\lambda(t^l_+) = r\Pi(t^l, l(t^l)) + f\big(\hat{l}(t^l_+)\big) - wl + \dot{l}(t^l_+)\lambda(t^l_+).$$

It can be interpreted that, on the leaving time, the benefits of postponed and immediate leave become equal. The left-hand side of the equation is the benefit of postponing the leave. By retarding the leave by dt, the firm receives bound instantaneous profits and the value of employment change according to bound dynamics. The right-hand side is the benefit of immediate leave. By obtaining the new state Π, the firm receives its return, and also instantaneous profits and the value of employment change both according to the unbound path.

The leaving behavior is summarized symmetrically to the entering case.

PROPOSITION 10: *At leaving time, production discontinuously decreases so that binding demand constraint beforehand becomes unbinding afterwards. Hiring discontinuously increases from zero to strictly positive. Costate variable λ jumps from zero to strictly positive. Demand dual μ jumps from strictly positive to zero. Time path of l kinks so that $\dot{l}$ jumps upwards.*

Figure 4b draws how the jump condition (29) and the terminal surface (22) determine the entering time. Different from ordinary optimization problems, the initial employment in the truncated problem after the leave is not given. The choice of t^l directly determines it according to the demand constraint (26). The jump condition (25) drawn as as a broken curve on the above plane simultaneously determines $\lambda(t^l_+)$ which corresponds to the hypothesis on λ_0. If the choice of t^l is correct, the dynamics of λ (6) provides the value of λ which satisfies the transversality condition if t^l is the last leaving time. If there is another entering to Λ^o, the dynamics satisfies the entering conditions described above at the next entering time. Figure 5 draws the kink in the employment path and the jump in the production frontier at the leaving time.



(a) Kink in employment path (b) Jump on production frontier

Figure 5. : Leaving behavior

VII. Weak labor hoarding

A labor asset model can have labor hoarding within business cycles since its demand depends on the value of the labor asset which does not necessarily meet the demand for the spot labor expense. In the present model with no firing cost, it only occurs in a weak sense that firing will not take place even when the decline in the spot labor demand is more than the natural separation. In this case, since the value of labor is positive, they are instead hired in the hiring sector. This is easily observed by an example. Figure 6 shows a periodic steady state of a

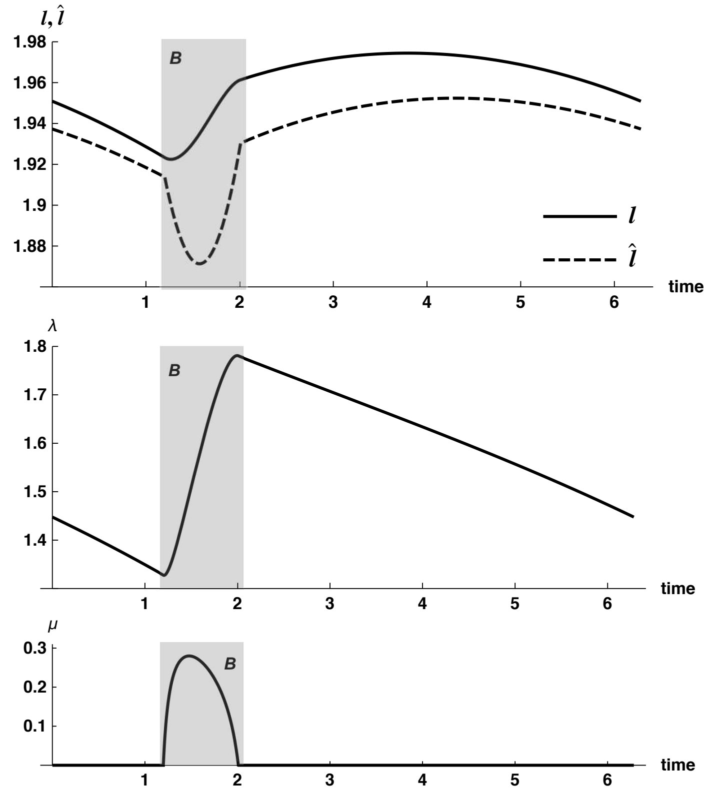


Figure 6. : Periodic steady state of the toy economy

toy economy in which the demand has period 2π and the level of demand falls below the unconstrained steady state only in $t \in (0.61, 2.53)$ in the principal domain $[0, 2\pi)$.[10] Studying the properties of periodic steady state is beneficial to close this type of optimization problem with infinite horizon since specifying external force y for infinite period is literally impossible and effects of faraway future is discounted anyway. Instead of keeping y open-ended, we can safely

[10]Note that this is a non-autonomous dynamical system. The periodicity is brought by that of the demand constraint potentially binding. The toy economy has the production function $f(\hat{l}) = 5\,\hat{l}^{3/4}$ and $g(\tilde{l}) = \tilde{l}^{3/4}$. Separation occurs at the rate 0.03. Discount rate is set to 0.05. The demand constraint is $y = 10 + 2\sin(t - \pi)$ and wage rate is constant at $w = 3$. Note that, even though demand constraint is binding only in some subperiod, it brings periodicity upon the whole optimal path. For the properties of periodic forced oscillation for linearized systems, see Kato, Naito and Shin (2005).

close it as a loop with sufficiently long period.[11] The optimal path binds to the constraint only in its subset, i.e. $t \in (1.05, 2.07) =: B$. As shown in the first figure, although $\hat{l} = f^{-1}(y)$ in B, employment begins to increase right after entering in B. Redundant labor is utilized to enforce the hiring sector in preparation for future increase of production. In B^c, the firm *unboundedly chooses* to operate in a lower production level than the overall unconstrained case in which the optimal employment is constant at $l = 2.17$. Note that, in the constraint binding phase, the increase of employment is driven by the improvement of labor value shown in the second figure. Since $\dot{\lambda} = (r+\sigma)\lambda - (1-\mu)f'(\hat{l}) + w$, the rise of μ in the binding phase more quickly improves the value of labor λ by adding external forces. Note that, in the unbinding phase, the external force that affects λ is only through the change of the marginal productivity of labor. Drastic improvement of labor value happens more easily in the binding phase in this sense. Also, note that the existence of a small period of the binding phase can affect the whole dynamics. In this example, the binding phase occupies only sixteen percent of the total period.

VIII. Strong labor hoarding and firing cost

On the other hand, if there exists firing cost, strong labor hoarding can arise in the sense that part of employment is put idle. Assume that there exists convex firing cost $\kappa(x)$ where $\kappa : \mathbb{R}_+ \to \mathbb{R}$, $\kappa', \kappa'' > 0$, $\kappa(0) = \kappa'(0) = 0$ and $\kappa'(+\infty) = +\infty$. This specification implicitly assumes that firing activity does not consume internal human resources. This would be approximately true if sufficient information on worker properties that is necessary for selection of firing target is already accumulated within everyday work, and if the main cost of firing is pecuniary compensation. We also impose a moderate assumption that $f'(\hat{l}) \geq w$ reflecting the bargaining outcome that the value of profits is strictly positive. We modify the objective function to $\Pi = \max_{\hat{l},x} \int_0^\infty \big(f(\hat{l}) - w\,l - \kappa(x) - c\big)\,\mathrm{e}^{-rt}\mathrm{d}t$, the labor transition to $\dot{l} = g(\tilde{l}) - \sigma l - x$ where $\tilde{l}$ is the employment in the hiring sector and add conditions $\tilde{l} \geq 0$ and $\hat{l} + \tilde{l} \leq l$ to allow for idle employment. Denote adjoint variables to the last two constraints by η and θ, respectively. The demand constraint is unchanged. Also, we can safely omit the constraint $\hat{l} \geq 0$ as far as $y > 0$ holds. Then, the first order conditions become

$$(1-\mu)\,f'(\hat{l}) = g'(\tilde{l})\,\lambda + \eta = \theta \tag{30}$$

$$x = \begin{cases} 0 & \text{if } \lambda \geq 0 \\ \kappa'^{-1}(-\lambda) & \text{if } \lambda \leq 0. \end{cases} \tag{31}$$

Costate dynamics is unchanged from (6). Different from the previous model, $\lambda < 0$ is required for firing to exist. Suppose $\lambda < 0$ so that $x > 0$. From (30) and $\mu \geq 0$, we get $\eta > 0$ and $\tilde{l} = 0$. If $\mu = 0$, (9) implies $\dot{\lambda} < 0$. If $\mu = 0$ continues

[11] This is true even when there is a long-run trend in y.

to hold, it finally violates the constraint $l \geq 0$. So, there must be a period that $\mu > 0$ holds until $\lambda \geq 0$ is achieved. Take such a period. Since $\mu > 0$, it requires $\dot{l} = \dot{y}/f'(f^{-1}(y))$ as far as we assume $\hat{l} = l$, which is generically impossible for a general function y. So, $\mu = 1 > 0$, $\theta = 0$ and thus the labor hoarding relation $\hat{l} = f^{-1}(y) \leq l$ generically holds.

IX. Implication to linear firing cost

The firing cost in the previous section is specified an exogenous factor represented by $\kappa(x)$. Such specification minimizes alteration of the basic model but may blur the actual origin of the cost. If the origin is viewed as that of human resources directed to firing activities, it would be natural to assume $\kappa(\cdot)$ to be convex with the same reason as the hiring cost. A formal specification for such firing cost should extend the basic model to add the firing sector. However, empirical studies such as Kramarz and Michaud (2010) show that there is quite different firing cost structure among different countries, implying that empirical total firing cost may need to keep functional form of $\kappa(\cdot)$ more general. Kramarz and Michaud (2010) points out firing regulations in France and finds linear firing cost from French data. If firing cost arising from the human resources is negligible, the total firing cost may be indeed almost linear. In such a case, the singularity results of the basic model apply, changing the definition of Λ^o being intervals of $\lambda = -k$ if $\kappa(x) = kx$. It assumes that firing cost does not accompany fixed cost. Then, the optimal condition for firing (8) changes to

$$x = \begin{cases} 0 & \text{if } \lambda > -k \\ [0, \bar{x}] & \text{if } \lambda = -k \\ \bar{x} & \text{if } \lambda < -k \end{cases}$$

for arbitrarily large $\bar{x}$, whereas other conditions are unchanged. It implies that there exists no-firing and no-hiring interval in λ, i.e. $-k < \lambda \leq 0$. In that interval, $\eta > 0$ and $\theta \geq 0$ hold in (30). The same arguments in the previous section apply and in general strong labor hoarding is observed in that interval.

X. Conclusion

The basic assumptions added in this model are the existence of variety both in goods and workers. Whereas variety of goods are observable, that of workers are not by nature of human ability which causes search in face of hiring, which combines monopolistic model in the output market with search in the labor market. Existence of convex hiring cost is critical for the existance of uncoordinated paths. It arises naturally from decreasing returns of individual hiring activity, which prohibits jumps to the steady state. Namely, the "transition" process matters. Moreover, sufficiently high degree of differentiation in output goods does not always allow simple transition to the unbounded steady state. Rather it confers a main role of determination of the whole path on coordinated expectation. In those cases, the transition process is not a transition any more.

Viewing the problem from the perspective of the maximum principle, the present model showed that the jump of costate variables can occur *only with control constraints* in singular problems. Jump condition of the state-constrained problems have been well-known. However, singularity can bring indifferentiability of state variables on boundary even when only control constraints are included in the problem, which truncates the problem into subperiods in a similar fashion to the state-constrained problems and brings about the costate jumps.

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