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Abstract

In order to formulate rigorously a perfect competition in markets R.J. Aumann introduced a mathematical model of an atomless exchange economy. W. Hildenbrand introduced a formulation of production economies into an atomless exchange economy. In its formulation each production unit has a positive measure. As a consequence, despite the fact that each production unit has influences over market prices, it takes prices as given. This paper introduces production units, each of which is atomless and naturally tied to atomless agents in the consumption sector, and a competitive equilibrium in such a production atomless economy is shown to exist.

1 Introduction

Production in a Large Economy

In the treatment of general economic equilibrium the traditional economic concept of perfect competition has played a fundamental role. The essential idea of this concept is that every individual economic agent has only a negligible influence on the outcome of collective activities, thus acting as "price-takers". A normal justification of this is that the number of economic agents is very large. As is well-known, R.J. Aumann [2,3] has given a precise formulation of this idea and introduced the fundamental concept of an atomless measure space of economic agents. He proved the existence of an equilibrium for a pure exchange economy with an atomless measure space of economic agents [3]. In an effort to analyze economies with production in a measure space of consumers W. Hildenbrand [10] has shown two approaches. One is a straight forward generalization of Arrow and Debreu's definition of

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a private ownership economy. Along with an atomless comsumption sector W. Hildenbrand considered an *a priori* given set of production units. All agents including these production units are considered to act as price-takers.

The other is to introduce concept of coalition production economies. In the analysis of the core of an economy it is necessary to know what production possibilities are available to each coalition. For this reason Hildenbrand introducded production into an Aumann continuum economy by specifying for every coalition a production possibility set. A coalition is a measurable set of consumers. A coalition production economy is defined to be a measure space of consumers along with characteristics of each consumer and with a coalition production set correspondence. Here again each coalition acts as a price-taker. W. Hildenbrand [10] proved the existence of (quasi-)equilibrium in such economies.

It is felt by this author that in such economies with production and a measure space of consumers there is a fundamental difficulty in the notion of competitive equilibria in economies. Both in private ownership economies and in coalition production economies, production units or coalitions which can actually produce have positive weights whereas each consumer has a zero weight in commodity markets. Thus it does not seem to be appropriate to assume that production units or coalitions with positive measure take prices as given because they do have positive influences over market transactions. It is a cynical fact that a model which purports to express the idea of perfect competition in an ideal way has a difficulty in justifying the price-taking behavior of firms.

One might still argue, however, that as the number of possibly producing firms being the number of coalitions is infinite in coalition production economies, the assumption of price-taking behavior on the part of firms is relevant. Nevertheless, we counter this argument since it is not the number of possibly producing agents, but the power of each agent exercised in the markets, which is crucial to the working of perfect competition.

Reformulation of Production in a Large Economy

I believe that there are primarily two ways to resolve the difficulty mentioned earlier. One is to abandon the assumption of price-taking behavior of producing agents. The other is to reduce decision-making units within a single producing agent to each primary economic agent so that atomless primary agents behave as price-takers. In order to make the meaning of these statements clearer, let us first describe institutional framework and working of economies that we are to consider.

The basic components of economies are *primary economic agents* that are described by an atomless positive (and finite) measure space. Fundamental information about these agents, *i.e.* information about their individual needs, tastes, and endowments, is given as initial data. In addition, information about technological dispersion among primary economic agents is given. We assume that this technological dispersion is such that in order to realize a positive amount of production of any commodity, grouping of primary economic agents, basic knowledge of technologies, their labor, entrepreneurial knowledge, and/or inputs of their endowments are essential, that is, in addition to necessary technological knowledge and various inputs, only groups of primary economic agents with positive measure can materially produce commodities. These groups of primary economic agents, when viewed as production units, may be called *"compound"* agents, or *"producing"* agents (i.e. *firms*), distinct from primary economic agents that cannot carry out "positive" productive activities. Now, if we regard these producing agents as single decision-making units, a natural way to formulate a production model with a measure space of primary economic agents is to let each producing agent to take into account the effect of its own decision to the entire economy.

As a first step toward introducing "imperfections" into the production sector of economies, we may consider a very simple product-distribution process. All the commodities produced by a producing agent are distributed directly to primary economic agents who participate in organizing that particular producing agent. An exchange market is formed among primary economic agents, where the price-taking behavior of each agent can be justified in a natural way since the measure space of primary economic agents is atomless. If we assume that each producing agent possesses the exact information of this market process and knows the equilibrium prices which may result from each output level of all the producing agents, then their decision-making would be as follows: Output composition and the level of output of each producing agent is determined so as to maximize its profit, given the output levels of other producing agents and taking account of the impact of its own decision on the level of equilibrium prices. Such an equilibrium concept is a combination of Cournot-Nash equilibrium in the production sector and usual competitive equilibrium in the consumption sector. In this formulation of production economies, the difficulty of specifying which producing agents produce which products can be avoided by taking a production vector of each producing agent as its decision variable.

If we regard decision units to be primary economic agents within each producing agent instead of each producing agent itself being a single decision unit, then we come to another formulation of a production model with a measure space of primary economic agents. It is the notion of a "production atomless economy" which we want to develop in this paper. This production model is described as follows: Although "actual" (positive) production can be realized only by a group of primary economic agents with positive measure, each primary economic agent is understood to have knowledge of how much it can contribute to material production of the group. Since each primary economic agent cannot influence the whole outcome, it takes prices as given and decides how it contributes to production so as to maximize its "profit" (*i.e.* infinitesimal profit) within the range of its possible contributions to productions. In other words, we start out our model of production economies by taking individualistic view point on the consumption side but take collective view point on the production side and reduce its decision-making problem on the production side to that of each individual so as to achieve symmetric treatment of both sides. We note that in a production atomless economy no single primary economic agent has influence on the whole situation, whereas the aggregate behavior of a "large" group (*i.e.* that with positive measure) of primary agents can change realized profits.

Ideas expressed here are formally stated in Section 3 of the paper. But before we go into the formulation of our model, it will be convenient to set up some mathematical notation used in the remainder of the paper.

2 Mathematical Notation and Convention

$\mathbb{N}$ denotes the set of positive integers $\{1, 2, ...\}$. $\mathbb{R}^\ell_+$ denotes the nonnegative orthant of the Euclidean space $\mathbb{R}^\ell$.

Given two sets E and F, a *correspondence* φ from E to F, written as $\varphi : E \to F$, is a function from E to the subsets of F. *It's graph* is given by

$$G(\varphi) = \{(x, y) \in E \times F \mid y \in \varphi(x)\}.$$

Let $(A, \mathcal{A}, \nu)$ be a measure space, *i.e.*, A is a set, $\mathcal{A}$ a σ-field of subsets of A, and ν a σ-additive measure defined on $\mathcal{A}$. A function f from A to $\mathbb{R}^\ell$ is *integrable* if $f^i \in L_1(A, \mathcal{A}, \nu)$ (L_1 for short) for each $i = 1, ..., \ell$, where $f = (f^1, ...f^\ell)$ and $f^i : A \to \mathbb{R}$. (For $L_1(A, \mathcal{A}, \nu)$ see Dunford and Schwartz [7].) Denote the ℓ-fold product of $L_1(A, \mathcal{A}, \nu)$ by $L^\ell_1(A, \mathcal{A}, \nu)$, and L^ℓ_1 for short; then, $f \in L^\ell_1(A, \mathcal{A}, \nu)$ if and only if $f : A \to \mathbb{R}^\ell$ is integrable. Here, as usual, we are referring to elements of $L_1(A, \mathcal{A}, \nu)$ as if they were functions and instead of using the symbol $[f]$ (an equivalence class of f) we simply write as f, since no confusions will result. Let $f = (f^1, ...f^\ell) \in L^\ell_1$. The integral of f, denoted by $\int_A f d\nu$, is $(\int_A f^1 d\nu, ..., \int_A f^\ell d\nu)$.

The *dual space* of the topological vector space $L_1(A, \mathcal{A}, \nu)$ (with the L_1 norm) is the vector space $L^*_1 = L^\infty$ whose elements are continuous linear functionals on L_1. The topology induced by the L_1 norm is denoted by τ. The L^*_1-topology on L_1, *i.e.* the weak topology on L_1 induced by L^*_1, is called the weak topology on L_1 and is denoted by τ_w. τ_w is weaker than τ, *i.e.* $\tau_w \subset \tau$. The ℓ-fold products of τ and τ_w are, denoted by τ^ℓ and τ^ℓ_w respectively.

Given a measure space $(A, \mathcal{A}, \nu)$, a correspondence $\varphi : A \to \mathbb{R}^\ell$ is ν-*measurable*, or simply, *measurable* if its graph $G(\varphi) = \{(a, x) \in A \times \mathbb{R}^\ell | \, x \in \varphi(a)\}$ belongs to the product σ-field $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^\ell)$ where $\mathcal{B}(\mathbb{R}^\ell)$ denotes the σ-field of Borel subsets in $\mathbb{R}^\ell$.

A correspondence $F : \mathcal{A} \to \mathbb{R}^\ell$ defined on a measurable space $(A, \mathcal{A})$ is *countably additive* (or σ-additive) if $F(\cup_{j=1}^\infty E_j) = \Sigma_{j=1}^\infty F(E_j)$ for every sequence $\{E_1, E_2, ...\}(= \{E_j\}_j)$ of mutually disjoint elements of $\mathcal{A}$. Here the sum $\Sigma_{j=1}^\infty S_j$ of subsets $S_1, S_2, ...$ of $\mathbb{R}^\ell$, consists of all the vectors s such that $s = \Sigma_{j=1}^\infty s_j$ for some series $\{s_j\}_{j \in \mathbb{N}}$ that is absolutely convergent and $s_j \in S_j$ for every $j \in \mathbb{N}$.

Now, given a correspndence $F : A \to \mathbb{R}^\ell$, let $\mathcal{L}_F$ be the set of all integrable functions from A to $\mathbb{R}^\ell$ such that for each $f \in \mathcal{L}_F$, $f(a) \in F(a)$ for ν-*a.e.* $a \in A$, *i.e.*,

$$\mathcal{L}_F = \left\{ f \in L_1^\ell(A, \mathcal{A}, \nu) | \, f(a) \in F(a) \ \nu\text{-}a.e. \ a \in A \right\}.$$

The *integral* of the correspodence $F : A \to \mathbb{R}^\ell$ is defined by

$$\int_A F d\nu = \left\{ \int_A f d\nu \middle| \, f \in \mathcal{L}_F \right\}.$$

Given a correspndence $\Gamma : \mathcal{A} \to \mathbb{R}^\ell$, a *Radon-Nikodym derivative* of Γ is a corresponcence $F : A \to \mathbb{R}^\ell$ such that

$$\Gamma(E) = \int_E F d\nu \ \text{ for every } E \in \mathcal{A}.$$

Let $\mathcal{N}$ be the class of null sets in $(A, \mathcal{A}, \nu)$, $i.e. E \in \mathcal{N}$ if and only if there exists N such that $E \subset N$ and $\nu(N) = 0$. The measure space $(A, \mathcal{A}, \nu)$ is said to be (algebraically) *separable* if the algebra $\mathcal{A}/\mathcal{N}$ is generated by a countable set of its elements.

3 Formal Model and Result

3.1 Measure Space of Primary Economic Agents

A basic component of economies is the set of *primary economic agents* A. They form a separable measure space $(A, \mathcal{A}, \nu)$ where $\mathcal{A}$ is a σ-field of subsets of A, and ν a σ-additive atomless positive measure on $\mathcal{A}$ with $\nu(A) = 1$. Points of A are primary economic agents, and subsets F of A which belong to $\mathcal{A}$ are called *coalitions* or *groups* of primary economic agents. The number $\nu(F)$ represents the fraction of the totality of primary economic agents belonging to the coalition F.

Let X be a ν-measurable correspondence of A into subsets of $\mathbb{R}^\ell$ (where $\mathbb{R}^\ell$ represents the *commoditity space*),*i.e.* the graph $G(X) = \{(a, x) \in A \times \mathbb{R}^\ell | x \in X(a)\}$ belongs to the product σ-field $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^\ell)$. The set $X(a)$ is interpreted as the consumption set of primary economic agent $a \in A$.

For every $a \in A$ let $\precsim_a$ denote the preference-indifference relation of primary economic agent a which is a complete preordering defined on $X(a)$ such that the preference-indifference correspndence $\precsim$: $A \to \mathbb{R}^\ell \times \mathbb{R}^\ell$ is measurable, *i.e.*

$$\{(a, x, y) \in A \times \mathbb{R}^\ell \times \mathbb{R}^\ell | \, x \precsim_a y\} \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^\ell) \otimes \mathcal{B}(\mathbb{R}^\ell) \, .$$

Finally, let ω be a ν-measurable function of A into $\mathbb{R}^\ell_+$, describing the distribution of *a priori* given resources.

An *atomless exchange economy* of the primary economic agents is specified by $\mathcal{E}_{ex} = [(A, \mathcal{A}, \nu), X, \precsim, \omega]$.

3.2 Individual and Collective Properties of Primary Economic Agents

Throught the paper we assume the following individual and collective properties of primary economic agents.

(C1) For each $a \in A, X(a) = \mathbb{R}^\ell_+$

(C2) For ν-*a.e.* $a \in A$, the preference-indifference relation $\precsim_a$ is continuous, convex, monotonic, and locally nonsatiated, *i.e.*, for every $y \in X(a)$ the sets $\{x \in X(a) | \, y \precsim_a x\}$ and $\{x \in X(a) | \, x \precsim_a y\}$ are closed ; for every $z \in X(a)$, the set $\{x \in X(a) | \, z \precsim_a x\}$ is convex; $x, y \in X(a)$, $x \leq y$ implies $y \succ_a x$ [1]; in each neighborhood U_z of $z \in X(a)$ there exists a vector $x \in X(a) \cap U_z$ such that $x \succ_a z$

(C3) $\omega(a) \gg 0$ for each $a \in A$, and $\displaystyle\int_A \omega d\nu$ is finite.

3.3 Producing Agents

Producing agents in an economy are by definition coalitions or groups of primary economic agents having a positive measure. That is, $F \in \mathcal{A}$ is called a producing agent if $\nu(F) > 0$. This definition is based upon a consideration that technologies are dispersed among the primary economic agents at the outset. It means that productions depend upon not only marketed inputs which are represented by the commodity space $\mathbb{R}^\ell$, but also on some limitted resources not explicitly introduced into the model. A typical case would be

[1] $y \succ_a x \iff [y \not\precsim_a x$ and $x \precsim_a y]$

that of resources which are not marketable since they are indivisible or not measurable in real terms.

Let $F \in \mathcal{A}$ be a producing agent. F is endowed with technology and nonmarketed resources which are provided by the participating primary economic agents $a \in F$. Now assume that the producing agent F can transform input $y^- \in \mathbb{R}^\ell_+$ of marketed commodities into output $y^+ \in \mathbb{R}^\ell_+$. Then the vector $y = y^+ - y^-$ is an admissible production of the producing agent F. $\mathbb{Y}(F)$ denotes the set of such admissible productions. Differently put, we are given a coalition production possibility correspondence $\mathbb{Y} : \mathcal{A} \to \mathbb{R}^\ell$ on the measure space $(A, \mathcal{A}, \nu)$ since technologies and nonomarketed resources of primary economic agents are distributed *a priori* among them. We assume that the correspondence $\mathbb{Y} : \mathcal{A} \to \mathbb{R}^\ell$ has the following properties:

(Y1) $0 \in \mathbb{Y}(F)$ for each $F \in \mathcal{A}$, *i.e.*, no group of agents is forced to produce.

(Y2) $\mathbb{Y} : \mathcal{A} \to \mathbb{R}^\ell$ is ν-continuous, that is, $\nu(F) = 0$ implies $\mathbb{Y}(F) = \{0\}$, *i.e.*, primary economic agents have to form a group having positive weight in the measure space of primary economic agents in order to produce commodities.

(Y3) $\mathbb{Y}(A)$ is closed in $\mathbb{R}^\ell$,*i.e.*, aggregate production possibilities are continuous.

(Y4) $\mathbb{Y}(A) \cap \mathbb{R}^\ell_+ = \{0\}$, *i.e.*, free production is not possible.

(Y5) $\mathbb{Y} : \mathcal{A} \to \mathbb{R}^\ell$ is countably additive, *i.e.*, technologies and nonomarketed resources are well dispersed among the primary economic agents so that a particular way of grouping production units in an economy does not give an advantage to any production unit.

3.4 Feasible Production Possibility Correspondence

Given a coalition production possibility correspondence $\mathbb{Y} : \mathcal{A} \to \mathbb{R}^\ell$ with properties (Y1)–(Y5), the *feasible production possibility correspondence* $\hat{\mathbb{Y}} : \mathcal{A} \to \mathbb{R}^\ell$ is defined in the following way: First, define $\hat{\mathbb{Y}}(A)$ by

$$\hat{\mathbb{Y}}(A) = \left\{ y \in \mathbb{Y}(A) \;\middle|\; y + \int_A \omega d\nu \geq 0 \right\}.$$

Then, for $F \in \mathcal{A}$ with $\nu(F) = 0$, set $\hat{\mathbb{Y}}(F) = \mathbb{Y}(F)$ $(= \{0\})$. To define $\hat{\mathbb{Y}}(F)$ for each $F \in \mathcal{A}$ with $\nu(F) > 0$, consider the set of measurable partitions $\mathcal{P}_F$ (not necessarily finite) of A such that $\mathcal{C} \in \mathcal{P}_F$ if and only if $\mathcal{C} = \{E_i\}_{i \in \mathbb{N}}$ with $E_i \in \mathcal{A}$ for each $i \in \mathbb{N}$, $\cup_{i=1}^\infty E_i = A$, $E_i \cap E_j = \emptyset$ if $i \neq j$, and $E_{i_0} = F$ for some i_0. Then $y_F \in \hat{\mathbb{Y}}(F)$, by definition, if there exists an absolutely convergent series $\{y_i\}_i$ and a partition $\{E_i\}_i \in \mathcal{P}_F$ such that $y_i \in \mathbb{Y}(E_i)$ for each $i \in \mathbb{N}, F = E_{i_0}$ for some i_0, $y_F = y_{i_0}$ and $y = \Sigma_{i=1}^{\infty} y_i \in \hat{\mathbb{Y}}(A)$.

The feasible production possibility corresponcence $\hat{\mathbb{Y}} : \mathcal{A} \to \mathbb{R}^\ell$ thus introduced is shown to satisyfy the following basic properties:

Lemma 3.1 *The feasible production possibility corresponcence* $\hat{\mathbb{Y}} : \mathcal{A} \to \mathbb{R}^\ell$ *is countably additive, and* $\hat{\mathbb{Y}}(A)$ *is closed and bounded in* $\mathbb{R}^\ell$*.*

Proof.

Countable Additivity: Let $\{E_i\}_i$ be a sequence of mutually disjoint elements of $\mathcal{A}$. We first consider the case where $\{E_i\}_i$ is a measurable partition of A, *i.e.* $\cup_{i=1}^{\infty} E_i = A, E_i \in \mathcal{A}$ for $i \in \mathbb{N}$, and $E_i \cap E_j = \emptyset$ if $i \neq j$.

Let $y \in \Sigma_{i=1}^{\infty} \hat{\mathbb{Y}}(E_i)$; then there exists an absolutely convergent series $\{y_i\}_i$ such that $y = \Sigma_{i=1}^{\infty} y_i$ and $y_i \in \hat{\mathbb{Y}}(E_i)$ for each $i \in \mathbb{N}$. By definition of $\hat{\mathbb{Y}}(E_i)$, then, $y = \Sigma_{i=1}^{\infty} y_i \in \hat{\mathbb{Y}}(A)$. This proves $\Sigma_{i=1}^{\infty} \hat{\mathbb{Y}}(E_i) \subset \hat{\mathbb{Y}}(A)$.

Let $y \in \hat{\mathbb{Y}}(A)$; then, since $\hat{\mathbb{Y}}(A) \subset \mathbb{Y}(A)$ and $\mathbb{Y}(A) = \Sigma_{i=1}^{\infty} \mathbb{Y}(E_i)$, there exists an absolutely convergent series $\{y_i\}_i$ such that $y = \Sigma_{i=1}^{\infty} y_i \in \hat{\mathbb{Y}}(A)$ and $y_i \in \mathbb{Y}(E_i)$ for every $i \in \mathbb{N}$. Then by definition of $\hat{\mathbb{Y}}(A)$ and $\hat{\mathbb{Y}}(E_i)$, $y_i \in \hat{\mathbb{Y}}(E_i)$ for each $i \in \mathbb{N}$. Thus, $y \in \Sigma_{i=1}^{\infty} \hat{\mathbb{Y}}(E_i)$. Hence, $\hat{\mathbb{Y}}(A) \subset \Sigma_{i=1}^{\infty} \hat{\mathbb{Y}}(E_i)$, therefore, we have $\hat{\mathbb{Y}}(A) = \Sigma_{i=1}^{\infty} \hat{\mathbb{Y}}(E_i)$.

Next we consider the case where $\{E_i\}_i$ is not a partition of A. Since $\mathcal{A}$ is a σ-field, $\cup_{i=1}^{\infty} E_i \in \mathcal{A}$. Put $E_0 = A \setminus \cup_{i=1}^{\infty} E_i$; then $E_0 \in \mathcal{A}$ and $\cup_{i=0}^{\infty} E_i = A$. Hence by using the previous result we obtain

$$\begin{aligned}\hat{\mathbb{Y}}(\cup_{i=0}^{\infty} E_i) &= \hat{\mathbb{Y}}(E_0 \cup (\cup_{i=1}^{\infty} E_i)) \\ &= \hat{\mathbb{Y}}(E_0) + \hat{\mathbb{Y}}(\cup_{i=1}^{\infty} E_i) \\ &= \hat{\mathbb{Y}}(E_0) + \Sigma_{i=1}^{\infty} \hat{\mathbb{Y}}(E_i).\end{aligned}$$

Therefore, one obtains

$$\hat{\mathbb{Y}}(\cup_{i=1}^{\infty} E_i) = \Sigma_{i=1}^{\infty} \hat{\mathbb{Y}}(E_i).$$

Hence we have shown that $\hat{\mathbb{Y}} : \mathcal{A} \to \mathbb{R}^\ell$ is countably additive.

Closedness of $\hat{\mathbb{Y}}(A)$: Let $y_n \in \hat{\mathbb{Y}}(A)$ for each $n \in \mathbb{N}$ and $y_n \to y_0$ in $\mathbb{R}^\ell$; then $y_n + \int_A \omega d\nu \geq 0$ for each $n \in \mathbb{N}$. Hence, one has

$$y_0 + \int_A \omega d\nu \geq 0.$$

Since $\mathbb{Y}(A)$ is closed, $y_0 \in \mathbb{Y}(A)$. And thus, $y_0 \in \hat{\mathbb{Y}}(A)$, and $\hat{\mathbb{Y}}(A)$ is closed.

Boundedness of $\hat{\mathbb{Y}}(A)$: Suppose it were not bounded; then, there would exist ε-sphere at 0 in $\mathbb{R}^\ell$, $S_\varepsilon = \{x \in \mathbb{R}^\ell | \, \|x\| < \varepsilon\}$[2], with $\varepsilon > 1$ such that

$$(\forall t \geq 1)\, (\exists y_t \in \hat{\mathbb{Y}}(A))\, y_t \notin tS_\varepsilon\,.$$

[2]If $x = (x_1, \ldots, x_\ell), \|x\| = \max_{i=1,\ldots,\ell} \, |x_i|$

Since $\mathbb{Y}(A) \cap \mathbb{R}^\ell_+ = \{0\}$ implies $\hat{\mathbb{Y}}(A) \cap \mathbb{R}^\ell_+ = \{0\}$, one must have $y_t \notin \mathbb{R}^\ell_+$ for all $t \geq 1$. Letting k to be a real number such that $k > \| \int_A \omega d\nu \|$, we set $s = \max\{k/\varepsilon, 1\}$. Then $(\forall t \geq s)\ y_t + \int_A \omega d\nu \not\geq 0$, and hence one obtains $(\forall t \geq s)\ y_t \notin \hat{\mathbb{Y}}(A)$, which contradicts the fact that $(\forall t)\ y_t \in \hat{\mathbb{Y}}(A)$. Therefore $\hat{\mathbb{Y}}(A)$ is bounded in $\mathbb{R}^\ell$.

□

Corollary 3.1 *The correspondence $\hat{\mathbb{Y}} : \mathcal{A} \to \mathbb{R}^\ell$ has a measarable Radon-Nikodym derivative with compact and convex values.*

Proof.
By Theorem 4.2 and Theorem 9.1 of Artstein [1], this is a consequence of the above Lemma.

□

By this result we can write

$$\hat{\mathbb{Y}}(A) = \int_A Y d\nu = \left\{ \int_A f d\nu \;\middle|\; f \in \mathcal{L}_Y \right\}$$

where $Y : A \to \mathbb{R}^\ell$ is a measurable, compact- and convex-valued correspondence defined on A.

3.5 Individualized Point of View in Production

Corollary 3.1 allows us to introduce individualized or primary-agent-based point of view to the production sector of economies in the following sense: The production structure in our model of economies has been specified by the production possibility set for each coalition of primary economic agents. Since the coalition production possibility correspondence is ν-continuous, as far as each producing agent is regarded as a single decision unit, individual primary economic agents play no role in the production sector in explicit way. However, a Radon-Nikodym derivative $Y : A \to \mathbb{R}^\ell$ of the feasible production possibility correspondence $\hat{\mathbb{Y}} : \mathcal{A} \to \mathbb{R}^\ell$ assigns to each primary economic agent $a \in A$ the set $Y(a) \subset \mathbb{R}^\ell$. Thus, it naturally permit us to regard each primary economic agent to have a pseudo-production possibility set $Y(a)$. It might be interpreted to represent the set of "possible contributions of each primary economic agent to the productions of the economy." This $Y(a)$, which we may call the production set of a primary agent a, corresponds to the consumption set $X(a)$ in the exchange side of the economy. In this way we can treat the production sector in a symmetric way as in the consumption-exchange sector where we started our analysis from individualized point of view.

We assume that each primary economic agent decides "how much it contributes to the production," *i.e.* each $a \in A$ selects a point in $Y(a)$, so as to maximize infinitesimal profit arizing from this action. Note in passing that in this way we deprive of any meaning attached to the coalition formation on the part of primary economic agents. Hence, although the production structure introduced by the coalition production possibility correspondence differs from the Walrasian structure, in a sense we come back to a Walrasian production structure. We call this economy *a production atomless economy*. An interesting and unsolved problem in a production model, however, is to explain the determination of structure of firms (or, in our terminology, the structure of producing agents). The only explication that the present model might give as regard to the structure of producing agents is the one concerning the possibly differing sets of primary economic agents, each producing separated sets of commodities.

3.6 Production Atomless Economies

Let $\mathcal{E}_{ex} = [(A, \mathcal{A}, \nu), X, \precsim, \omega]$ be an atomless exchange economy having the properties (C1)–(C3). An integrable function $f : A \to \mathbb{R}^\ell$ is called a *consumption plan* if $f \in \mathcal{L}_X$.

Definition 1 *An ordered pair (f, p) of a consumption plan $f \in \mathcal{L}_X$ and a price vector $p \in \mathbb{R}^\ell_+$, $p \neq 0$, is called a* quasi-equilibrium *for the atomless exchange economy $\mathcal{E}_{ex}$ if f and p satisfy*

1. *$f(a)$ is a greatest element for $\precsim_a$ in the budget set $B(a,p) = \{x \in X(a) |\, p \cdot x \leq p \cdot \omega(a)\}$ and/or $\inf p \cdot X(a) = p \cdot \omega(a)$, ν-a.e. $a \in A$.*
2. $\displaystyle\int_A f d\nu = \int_A \omega d\nu.$

A quasi-equilibrium $(f^*, p^*) \in \mathcal{L}_X \times \mathbb{R}^\ell_+$ for $\mathcal{E}_{ex}$ is actually an equilibrium if ν-a.e. $a \in A$, either (i) $\inf p^* \cdot X(a) < p^* \cdot \omega(a)$, *i.e.*, with respect to an equilibrium price vector, the wealth is strictly greater than the infinimum compatible with the consumption set, or (ii) $\inf p^* \cdot X(a) = p^* \cdot \omega(a)$ implies that all vectors in $B(p^*, a)$ are equivalent for $\precsim_a$.

In an atomless exchange economy $\mathcal{E}_{ex}$ with monotonic preferences and where $X(a) = \mathbb{R}^\ell_+$ and $\int_A \omega d\nu \gg 0$, (ii) holds for ν-a.e. $a \in A$. In this case a price vector which belongs to a quasi-equilibrium is strictly positive and consequently,

$$0 = \int \inf p^* \cdot X(a) = p^* \cdot \omega(a) \text{ implies } B(p^*, a) = \{0\}.$$

Now define a *production atomless strategy* $\hat{y}$ to be an integrable function $\hat{y} : A \to \mathbb{R}^\ell$ such that $\hat{y}(a) \in Y(a)$ for ν-a.e. $a \in A$. A production atomless strategy $\hat{y}$ belongs to $L_1^\ell(A, \mathcal{A}, \nu)$. By the notation introduced in earlier section, the set of production atomless strategies is given by $\mathcal{L}_Y$, *i.e.*

$$\mathcal{L}_Y = \left\{ \hat{y} : A \to \mathbb{R}^\ell \;\middle|\; \hat{y} \in \mathrm{L}_1^\ell(A, \mathcal{A}, \nu), \text{ and } \hat{y}(a) \in Y(a) \text{ for } \nu\text{-a.e. } a \in A \right\}.$$

We endow $\mathcal{L}_Y$ with the subspace topology of τ_w^ℓ.

For each $\hat{y} \in \mathcal{L}_Y$ consider the corresponding atomless exchange economy $\mathcal{E}_{ex}(\hat{y}) = [(A, \mathcal{A}, \nu), X, \prec, \omega + \hat{y}]$. Suppose for the rest of this paragraph that the atomless exchange economy for each $\hat{y} \in \mathcal{L}_Y$ defines a function $p : \mathcal{L}_Y \to \mathbb{R}_+^\ell$ such that $p(\hat{y})$ is the equilibrium price vector for $\mathcal{E}_{ex}(\hat{y})$. Then, given $\hat{y} : A \to \mathbb{R}^\ell$, $\hat{y} \in \mathcal{L}_Y$, we define

$$r(a, y_a, \hat{y}) = p(\hat{y}) \cdot y_a \text{ for each } a \in A \text{ and } y_a \in Y(a)\,.$$

Definition 2 *A pair $(\hat{y}^*, p(\hat{y}^*))$ of a production atomless strategy $\hat{y}^* \in \mathcal{L}_Y$ and an equilibrium price vector $p(\hat{y}^*)$ for $\mathcal{E}_{ex}(\hat{y}^*)$, is called an equilibrium of the production atomless economy if, for ν-a.e. $a \in A$, $r(a, \hat{y}^*(a), \hat{y}^*) \geq r(a, y_a, \hat{y})$ for any $\hat{y} \in \mathcal{L}_Y$ and any $y_a \in Y(a)$.*

3.7 Atomless Exchange Equilibrium Prices

For each $\hat{y} \in \mathcal{L}_Y$, let $p(\hat{y})$ be the set of equilibrium prices for the atomless exchange economy $\mathcal{E}_{ex}(\hat{y})$. We know that under the assumptions (C1)–(C3), the atomless exchange economy $\mathcal{E}_{ex} = [(A, \mathcal{A}, \nu), X, \prec, \omega]$ has an equilibrium so that $p(0) \neq \emptyset$. Now, since $0 \in Y(a)$ for each $a \in A$, for ν-a.e. $a \in A$ $p(\hat{y}) \cdot \hat{y}(a) \geq 0$. As $\omega(a) \in \mathbb{R}_{++}^\ell$, we obtain

$$\inf p(\hat{y}) \cdot X(a) < p(\hat{y}) \cdot \omega(a) \leq p(\hat{y}) \cdot (\omega(a) + \hat{y}(a)) \text{ for } \nu\text{-a.e. } a \in A.$$

But a quasi-equilibrium exists for $\mathcal{E}_{ex}(\hat{y})$ for each $\hat{y} \in \mathcal{L}_Y$ under the assumptions (C1)–(C3); therefore, $p(\hat{y}) \neq \emptyset$ for each $\hat{y} \in \mathcal{L}_Y$.

Lemma 3.2 *The atomless exchange equilibrium price correspondence $p : \mathcal{L}_Y \to \mathbb{R}_+^\ell$ has a closed graph.*

Proof.

Let $\hat{y}_n \to \hat{y}_0$ in the τ_w^ℓ-topology with $\hat{y}_n \in \mathcal{L}_Y$ for each $n \in \mathbb{N}$ and for $\hat{y}_0 \in \mathcal{L}_Y$, and $p_n \to p_0 \in \Delta \subset \mathbb{R}_+^\ell$ such that $p_n \in p(\hat{y}_n)$ for each $n \in \mathbb{N}$.[3] Let (f_n, p_n) be an equilibrium for the atomless exchange economy $\mathcal{E}_{ex}(\hat{y}_n)$.

Then, (f_n, p_n) has the following properties for each $n \in \mathbb{N}$:

$$f_n(a) \in D(a, p_n) \text{ for } \nu\text{-a.e. } a \in A, \tag{1}$$

[3]The use of a sequence here is justified later. (See Remark 4.1 in Section 4.)

where

$$D(a,p_n) = \{x \in B(a,p_n) |\ z \precsim_a x \ for\ every\ z \in B(a,p_n)\}\,, \qquad (2)$$

and

$$\int_A f_n d\nu = \int_A (\omega + \hat{y}_n) d\nu\,. \qquad (3)$$

Now, $\hat{y}_n \to \hat{y}_0$ in the τ_w^ℓ-topology implies $\int_A \hat{y}_n d\nu \to \int_A \hat{y}_0 d\nu$ in $\mathbb{R}^\ell$, and thus we have

$$\lim_n \int_A (\omega + \hat{y}_n) d\nu = \int_A (\omega + \hat{y}_0) d\nu\,.$$

Therefore, by(3) $\lim_n \int_A f_n d\nu$ exists and

$$\lim_n \int_A f_n d\nu = \int_A (\omega + \hat{y}_0) d\nu\,. \qquad (4)$$

Then by Fatou's lemma in several dimensions (see Hildenbrand and Mertens [11] or Schemidler [13]) there exists an integrable function $f : A \to \mathbb{R}^\ell$ such that

$$f(a) \in \mathrm{adh}\,(f_n(a)) \text{ for } \nu\text{-a.e. } a \in A\ ^4, \qquad (5)$$

and

$$\int_A f d\nu \leq \int_A (\omega + \hat{y}_0) d\nu\,. \qquad (6)$$

Since the graph of demand correspondence D is closed [5], $\lim_n p_n = p_0$ implies

$$\mathrm{adh}\,(f_n(a)) \subset D(a,p_0) \neq \emptyset \text{ for } \nu\text{-a.e. } a \in A\,. \qquad (7)$$

(7) and the monotonicity of the preference relations imply that $p_0 \gg 0$.

Next we want to show that any function $f : A \to \mathbb{R}^\ell$ with property (5) in fact satisfies the equality of supply and demand, *i.e.*,

$$\int_A f d\nu = \int_A (\omega + \hat{y}_0) d\nu\,. \qquad (8)$$

Indeed (5) implies together with (7) and the monotonicity of the preference relations that

$$p_0 \cdot f(a) = p_0 \cdot (\omega(a) + \hat{y}_0(a)) \text{ for } \nu\text{-a.e. } a \in A\,.$$

[5]See for example, W. Hildenbrand,"On Economies with many Agents," Appendix A, *Journal of Economic Theory, 2*, 1970, pp.182-183.

Thus, we have

$$p_0 \cdot \int_A f d\nu = p_0 \cdot \int_A (\omega + \hat{y}_0) d\nu \,,$$

and hence (8) follows from the fact that we have

$$p_0 \gg 0, \quad \text{and}$$
$$\int_A f d\nu \leq \int_A (w + \hat{y}_0) d\nu \,.$$

Then, by using the corollary of Hildenbrand and Mertens ([11], p.154), we obtain:

$$\text{The sequence } (f_n)_n \text{ is relative } \tau_w^\ell\text{-compact,} \tag{9}$$

and

$$\begin{array}{c}\text{every } \tau_w^\ell\text{-adherent point } f \text{ of } (f_n)_n \text{ has the property that} \\ f(a) \in D(a, p_0) \text{ for } \nu\text{-a.e. } a \in A \,,\end{array} \tag{10}$$

as the convexity of preferences and (7) imply

$$\text{co adh}\,(f_n(a)) \subset D(a, p_0) \,.^6$$

Since by (9) and by the Eberlein-Smulian's theorem[7] one can get τ_w^ℓ-convergent subsequence of $(f_n)_n$, it follows from (10) that we have $p_0 \in p(\hat{y}_0)$. We thus conclude that $p : \mathcal{L}_Y \to \mathbb{R}_+^\ell$ has a closed graph.

□

We would like to place some assumptions on production atomless economies.

Assumption I *$r(\cdot, \cdot, \hat{y})$ is ν-measurable for each $\hat{y} \in \mathcal{L}_Y$.*

Assumption II *The set $p(\hat{y})$ contains at most one element for each $\hat{y} \in \mathcal{L}_Y$.*

Because of Assumption II we now regard the correpondence $p : \mathcal{L}_Y \to \mathbb{R}^\ell$ as a function $p : \mathcal{L}_Y \to \mathbb{R}^\ell$. Moreover, we have:

Corollary 3.2 *$p : \mathcal{L}_Y \to \mathbb{R}^\ell$ is continuous.*

---

[6] co S denotes the convex bull of S.

[7] See, for example, Dunford and Schwartz [7, pp.430–433].

This is an immediate consequence of Lemma 3.2.

Given an atomless exchange economy $\mathcal{E}_{ex} = [(A, \mathcal{A}, \nu), X, \precsim, \omega]$ and a coalition production possibility correspondence $\mathbb{Y} : \mathcal{A} \to \mathbb{R}^\ell$ satisfying (C1)–(C3) and (Y1)–(Y5), we call $\mathcal{E}_{pro} = [(A, \mathcal{A}, \nu), p, Y]$ a *production atomless economy* where p is the atomless exchange equilibrium price correspondence $p : \mathcal{L}_Y \to \mathbb{R}^\ell_+$ and $Y : A \to \mathbb{R}^\ell$ is a Radon-Nicodym derivative of $\mathbb{Y}$.

We now state the main theorem of this paper.

Theorem 1 *There exists an equilibrium of a production atomless economy $\mathcal{E}_{pro} = [(A, \mathcal{A}, \nu), p, Y]$ under the Assumption I and II.*

4 Proof of the Theorem

Lemma 4.1 *$\mathcal{L}_Y$ is a convex and compact set in $L^\ell_1(A, \mathcal{A}, \nu)$ with the τ^ℓ_w-topology.*

Proof.

Convexity: Let $f = (f^1, ..., f^\ell) \in \mathcal{L}_Y$, $g = (g^1, ..., g^\ell) \in \mathcal{L}_Y$, and $\alpha + \beta = 1$ with $\alpha, \beta \geq 0$. Clearly, $\alpha f + \beta g \in L^\ell_1$, because $\alpha f^j + \beta g^j \in L_1$ for each $j = 1, ..., \ell$, by the linearity of L_1. Since $\int_A f d\nu \in \hat{\mathbb{Y}}(A)$, and $\int_A g d\nu \in \hat{\mathbb{Y}}(A)$, the convexity of $\hat{\mathbb{Y}}(A)$ implies that $\alpha \int_A f d\nu + \beta \int_A g d\nu = \int_A (\alpha f + \beta g) d\nu \in \hat{\mathbb{Y}}(A)$. Now, $f \in \mathcal{L}_Y$ implies that $f(a) \in Y(a)$ for each $a \in A \setminus N_1$ with $\nu(N_1) = 0$. Similarly, $g(a) \in Y(a)$ for each $a \in A \setminus N_2$ with $\nu(N_2) = 0$. Hence $f(a)$ and $g(a)$ both belong to $Y(a)$ for each $a \in A \setminus N$ with $\nu(N) = 0$ where $N = N_1 \cup N_2$. Together with the convexity of $Y(a)$ this implies that $\alpha f(a) + \beta g(a) \in Y(a)$ for ν-a.e. $a \in A$. Therefore $\alpha f + \beta g \in \mathcal{L}_Y$. This shows that $\mathcal{L}_Y$ is convex.

Now let $\mathcal{L}_{Y^j}$ denote the projection of $\mathcal{L}_Y$ on the j-th factor space, *i.e.*, if $pr_j : L^\ell_1 \to L_1$ is the j-th projection map, then $\mathcal{L}_{Y^j} = pr_j(\mathcal{L}_Y)$. Also, let $Y^j(a)$ denote the projection of $Y(a)$ on the j-th factor space, so that if $\pi_j : \mathbb{R}^\ell \to \mathbb{R}$ is the j-th projection map, $Y^j(a) = \pi_j(Y(a))$.

To show $\mathcal{L}_{Y^j}$ is τ-bounded for each $j = 1, ..., \ell$: Given a basic neighborhood $V_\varepsilon = \{f \in L_1 | \int_A |f| d\nu < \varepsilon\}$ at 0 with $\varepsilon > 0$, we shall show that there exists $s > 0$ such that $t \geq s$ implies $tV_\varepsilon \supset \mathcal{L}_{Y^j}$.

First, define a map $\varphi^j : A \to \mathbb{R}^\ell$ by $\varphi^j(a) = \max\{|y| \mid y \in Y^j(a)\}$. Put $S = \{x \in \mathbb{R}^\ell | \, |x| < c\}$. The measurability of Y implies that of Y^j. Since Y^j is compact-valued in $\mathbb{R}$ and $\mathbb{R}$ is separable, $[G(Y^j) \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R})]$ is equivalent to $\{a \in A \mid Y^j(a) \subset S\} \in \mathcal{A}$ for each open set S in $\mathbb{R}$. (See Debreu [5], pp. 358-359.) Hence, we have $\{a \in A \mid \varphi^j(a) < c\} = \{a \in A \mid Y^j(a) \subset S\} \in \mathcal{A}$. Therefore φ^j is ν-measurable.

As Y^j is compact-valued in $\mathbb{R}$, φ^j is bounded. Since φ^j is nonnegative, the finiteness of ν implies $\varphi^j \in L_1(A, \mathcal{A}, \nu)$. Now, let $g \in \mathcal{L}_{Y^j}$; then $\int_A |g| d\nu \leq \int_A \varphi^j d\nu$. Take some $c \in \mathbb{R}$ such that $\int_A \varphi^j d\nu < c$, and set $s = \frac{c}{\varepsilon}$. Then, one has $\varphi^j \in tV_\varepsilon$ for all $t \geq s$. Thus $g \in tV_\varepsilon$ for all $t \geq s$. Therefore $\mathcal{L}_{Y^j}$ is τ-bounded. This argument holds for any choice of j. Hence, $\mathcal{L}_{Y^j}$ is τ-bounded for each $j = 1, ..., \ell$.

To show $\mathcal{L}_{Y^j}$ is τ-closed for each $j = 1, ..., \ell$: We first show that $\mathcal{L}_Y$ is τ^ℓ-closed. Let $(f_n)_n$ be a sequence such that $f_n \in \mathcal{L}_Y$ for each $n \in \mathbb{N}$ and $f_n \to f$ in the τ^ℓ-topology. We want to show $f \in \mathcal{L}_Y$. Write $f_n = (f_n^1, ..., f_n^\ell)$, each $n \in \mathbb{N}$, and $f = (f^1, ..., f^\ell)$. Then for each $j = 1, ..., \ell$, $f_n^j \to f^j$. Clearly, $f^j \in L_1$ for each j , as L_1 space is complete. Hence, $f \in L_1^\ell$.

Since the measure ν is finite, the convergence in L_1 implies the existence of a subsequence which converges ν-a.e. pointwise. Choose a subsequence of the sequence $(f_n)_n$ in the following way: Let $(f_k^{(1)})_k$ be a subsequence of $(f_n)_n$ such that $f_k^{(1)} \to f^1$, ν-a.e.. Then, choose a subsequence $(f_k^{(1,2)})_k$ of $(f_k^{(1)})_k$ so that $f_k^{(1,2)} \to f^2$, ν-a.e., and so on. We thus obtain a subsequence $(f_k^{(1,...,\ell)})_k$ of the sequence $(f_n)_n$ such that, $f_k^{(1,...,\ell)} \to f^j$, ν-a.e., for each $j = 1, ..., \ell$. Hence, $f_k^{(1,...\ell)}(a) \to f(a)$ for ν-a.e. $a \in A$. But as $Y(a)$ is closed for each $a \in A$ and $f_k^{(1,...,\ell)}(a) \in Y(a)$ for ν-a.e. $a \in A$, $f(a) \in Y(a)$, ν-a.e. $a \in A$. Thus $f \in \mathcal{L}_Y$, and $\mathcal{L}_Y$ is τ^ℓ-closed.

We proceed to show $\mathcal{L}_{Y^j}$ is τ-closed for each $j = 1, ..., \ell$. Let us fix the coordinate j. Let $(g_n^j)_n$ be a sequence such that $g_n^j \to g^j$ in the τ-topology and $g_n^j \in \mathcal{L}_{Y^j}$. To show $g^j \in \mathcal{L}_{Y^j}$. Since $\mathcal{L}_Y \neq \emptyset$ (by Corollary 3.1), pick $f_n \in {pr_j}^{-1}({g_n}^j) \cap \mathcal{L}_Y$ for each $n \in \mathbb{N}$ with $f_n = (f_n^1, ..., f_n^\ell)$ where $f_n^j = g_n^j$. Since $\mathcal{L}_{Y^i}$ is τ-bounded for each i and that τ is a metric topology, we can choose a convergent subsequence in the following way: Let $(f_n^{(1)})_n$ be a subsequence of the sequence $(f_n)_n$ such that $f_n^{(1)_1}$ converges in τ to, say, f^1. Then, from the subsequence $(f_n^{(1)})_n$ we choose a subsequence $(f_n^{(1,2)})_n$ so that $f_n^{(1,2)_2}$ converges in τ to say, f^2. We repeat this process to obtain a sequence $(f_k^{(1,...,\ell)})_k$ which is a subsequence of the sequence$(f_n)_n$ such that for each $i = 1, ..., \ell$, $f_k^{(1,...,\ell)_i} \to f^i$ in the τ topology. Clearly, $f^j = g^j$. Set $f = (f^1, ..., f^\ell)$. Then, $f_k^{(1,...,\ell)} \to f$ in τ^ℓ. Since $\mathcal{L}_Y$ is closed by the earlier argument, we get $f \in \mathcal{L}_Y$. Therefore, $f^j = g^j \in \mathcal{L}_{Y^j}$, showing that $\mathcal{L}_{Y^j}$ is τ-closed. Since this argument holds for any choice of a coordinate j, $\mathcal{L}_{Y^j}$ is τ-closed for each $j = 1, ..., \ell$.

To show $\mathcal{L}_Y$ is τ_w^ℓ-compact: Since τ is a metric topology, $\mathcal{L}_{Y^j}$ is τ-compact by the above arguments. As τ_w is weaker than τ so that $\tau_w \subset \tau$, $\mathcal{L}_{Y^j}$ is τ_w-compact. Since this holds for each $j = 1, ..., \ell$, by Tychonoff's theorem, $\mathcal{L}_Y$ is τ_w^ℓ-compact.

□

Next, given $\hat{y} \in \mathcal{L}_Y$, define for each $a \in A$

$$\mu_a(\hat{y}) = \left\{ y^* \in Y(a) \,\middle|\, (\forall y \in Y(a))\; p(\hat{y}) \cdot y^* \geq p(\hat{y}) \cdot y \right\}.$$

Then we have :

Lemma 4.2 $\mu_a(\hat{y})$ *is convex and nonempty.*

Proof.
Let $y_1, y_2 \in \mu_a(\hat{y})$, $\alpha, \beta \geq 0$, and $\alpha + \beta = 1$. Then, for any $y \in Y(a)$,

$$\begin{aligned} p(\hat{y})(\alpha y_1 + \beta y_2) &= \alpha p(\hat{y}) \cdot y_1 + \beta p(\hat{y}) \cdot y_2 \\ &\geq \alpha p(\hat{y}) \cdot y + \beta p(\hat{y}) \cdot y \\ &= p(\hat{y}) \cdot y \end{aligned}$$

Hence, $\alpha y_1 + \beta y_2 \in \mu_a(\hat{y})$ as we also have $\alpha y_1 + \beta y_2 \in Y(a)$ by the convexity of $Y(a)$.

$\mu_a(\hat{y}) \neq \emptyset$ follows from the fact that $r(a, \cdot, \hat{y})$ is continuous and $Y(a)$ is compact in $\mathbb{R}^\ell$.

□

Lemma 4.3 $G(\mu_a) = \{(\hat{y}, y) \in \mathcal{L}_Y \times Y(a) \mid y \in \mu_a(\hat{y})\}$ *is closed in* $\mathcal{L}_Y \times Y(a)$.

Proof.
Let $(\hat{y}_\lambda, y_\lambda)_{\lambda \in \Lambda}$ be a net in $G(\mu_a)$ such that $y_\lambda \in \mu_a(\hat{y}_\lambda)$ and $(\hat{y}_\lambda, y_\lambda) \to (\hat{y}_0, y_0)$. To show $(\hat{y}_o, y_0) \in G(\mu_a)$. As $Y(a)$ is closed and $(y_\lambda)_\lambda$ is a net in $Y(a)$ with $y_\lambda \to y_0$, one has $y_0 \in Y(a)$.

$y_\lambda \in \mu_a(\hat{y}_\lambda)$ means that $p(\hat{y}_\lambda) \cdot y_\lambda \geq p(\hat{y}_\lambda) \cdot y$ for any $y \in Y(a)$. Since the continuity of the map $p : \mathcal{L}_Y \to \mathbb{R}^\ell_+$ implies that of $r(a, \cdot, \cdot)$, the inequality holds at the limit, so that one has, for any $y \in Y(a)$, $p(\hat{y}_0) \cdot y_0 \geq p(\hat{y}_0) \cdot y$. Thus, $y_0 \in \mu_a(\hat{y}_0)$ and $(\hat{y}_0, y_0) \in G(\mu_a)$. It follows that $G(\mu_a)$ is closed in $\mathcal{L}_Y \times Y(a)$.

□

We now define a correspondence $\mu : \mathcal{L}_Y \to \mathcal{L}_Y$ by

$$\mu(\hat{y}) = \left\{ \hat{y}^* \in \mathcal{L}_Y \,\middle|\, \hat{y}^*(a) \in \mu_a(\hat{y}) \text{ for } \nu\text{-a.e. } a \in A \right\},$$

and shows that it has the following property:

Lemma 4.4 *For each* $\hat{y} \in \mathcal{L}_Y$, $\mu(\hat{y})$ *is convex and nonempty.*

Proof.

Convexity: Let $\hat{y}_1, \hat{y}_2 \in \mu(\hat{y})$; then $\hat{y}_1(a) \in \mu_a(\hat{y})$ for $a \in A \setminus N_1$ with $\nu(N_1) = 0$, and $\hat{y}_2(a) \in \mu_a(\hat{y})$ for $a \in A \setminus N_2$ with $\nu(N_2) = 0$. Therefore both $\hat{y}_1(a)$ and $\hat{y}_2(a)$ belong to $\mu_a(\hat{y})$ for $a \in A \setminus N$ with $N = N_1 \cup N_2$ and $\nu(N) = 0$. Since $\mu_a(\hat{y})$ is convex for each $a \in A$, for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$, we have $\alpha\hat{y}_1(a) + \beta\hat{y}_2(a) \in \mu_a(\hat{y})$ for ν-a.e. $a \in A$. Also, the convexity of $\mathcal{L}_Y$ implies $\alpha\hat{y}_1 + \beta\hat{y}_2 \in \mathcal{L}_Y$. Thus, $\alpha\hat{y}_1 + \beta\hat{y}_2 \in \mu(\hat{y})$, showing $\mu(\hat{y})$ is convex.

Nonemptiness: Given $\hat{y} \in \mathcal{L}_Y$, define a correspondence $Z : A \to \mathbb{R}^\ell$ by $Z(a) = \mu_a(\hat{y})$ fpr each $a \in A$. For $j = 1, ..., \ell$, let $\varphi^j : A \to \mathbb{R}^\ell$ be the map defined in the proof of Lemma 4.1. Set $\varphi = (\varphi^1, ..., \varphi^\ell)$. Then, it can be easily checked that $\varphi : A \to \mathbb{R}^\ell$ is an integrable function and, for each $y \in \mu_a(\hat{y})$, $\|y\| \leq \|\varphi(a)\|$. Hence, $Z : A \to \mathbb{R}^\ell$ is integrably bounded.

Next, we show the measurability of the graph

$$G(Z) = \{(a, y) \in A \times \mathbb{R}^\ell \mid y \in Z(a)\}.$$

Define a function $v : A \to \mathbb{R}^\ell$ by $v(a) = \max\{y \in Y(a) | \, r(a, y, \hat{y})\}$. Since the correspondence $Y : A \to \mathbb{R}^\ell$ is measurable, $v : A \to \mathbb{R}$ is measurable; for, if we let S be the open set $\{x \in \mathbb{R}^\ell | \, p(\hat{y}) \cdot x < c\}$, one has $\{a \in A | \, v(a) < c\} = \{a \in A | \, Y(a) \subset S\}$ which belongs to $\mathcal{A}$ as $Y : A \to \mathbb{R}^\ell$ is compact-valued and $\mathbb{R}^\ell$ is separable (Debreu [5], (5.8), p.363]).

Now, $Z(a) = \{y^* \in Y(a) \mid r(a, y*, \hat{y}) = v(a)\}$. Since $r(a, \cdot, \hat{y})$ is continuous, $r(\cdot, \cdot, \hat{y})$ and $v(\cdot)$ are ν-measurable, and $\mathbb{R}^\ell$ is complete and separable, the graph $G(Z)$ of the correspondence $Z : A \to \mathbb{R}^\ell, \{(a, x) \in A \times \mathbb{R}^\ell \mid r(a, x, \hat{y}) = v(a)\}$ belongs to $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^\ell)$ (Debreu [5, (4)-(5), pp.360-361]).

Now that we established integrable boundedness and measurability of the correspondence $Z : A \to \mathbb{R}^\ell$, we can use Aumann's Theorem 2 (Aumann [4, p.2]) to obtain

$$\int_A Z d\nu = \left\{ \int_A \eta d\nu \;\middle|\; \eta \in L_1^\ell(A, \mathcal{A}, \nu) \text{ and } \eta(a) \in Z(a) \text{ for } \nu\text{-a.e. } a \in A \right\} \neq \emptyset \, .$$

This means that there exists $\eta : A \to \mathbb{R}^\ell$ such that $\eta \in L_1^\ell(A, \mathcal{A}, \nu)$ and $\eta(a) \in \mu_a(\hat{y})$ for ν-a.e. $a \in A$. It follows that $\eta \in \mu(\hat{y})$, showing $\mu(\hat{y}) \neq \emptyset$.

□

Remark 4.1 Before we proceed to the next lemma, we would like to note the following: Since the measure space $(A, \mathcal{A}, \nu)$ is (algebraically) separable, $L_1(A, \mathcal{A}, \nu)$ is separable (with the τ-topology). Now $\mathrm{pr}_j(\mathcal{L}_Y)$ is a weakly compact set by Lemma 4.1. This implies together with separability of $L_1(A, \mathcal{A}, \nu)$ that the weak topology of $pr_j(\mathcal{L}_Y)$ is metrizable (Dunford and Schwartz [Theorem V.6.3, p.434]). Hence so is the subspace topology on $\mathcal{L}_Y$

induced by τ_w^ℓ. This implies that $\mathcal{L}_Y$ with the subspace topology of τ_w^ℓ is first countable. Therefore, we can characterize closed sets in $\mathcal{L}_Y$ with the relative τ_w^ℓ-topology by using sequences.

□

Lemma 4.5 *The graph of* $\mu : \mathcal{L}_Y \to \mathcal{L}_Y$, $G(\mu) = \{(\hat{y}, \hat{\hat{y}}) \in \mathcal{L}_Y \times \mathcal{L}_Y \mid \hat{\hat{y}} \in \mu(\hat{y})\}$ *is closed.*

Proof.

Let $\hat{y}_n \to \hat{y}_0$, $\hat{\hat{y}}_n \to \hat{\hat{y}}_0$ (both in the τ_w^ℓ-topology), and, for each $n \in \mathbb{N}$, $\hat{\hat{y}} \in \mu(\hat{y})_n$. We are to show $\hat{\hat{y}}_0 \in \mu(\hat{y}_0)$. Suppose $\hat{\hat{y}}_0 \notin \mu(\hat{y}_0)$; then, there are $\tilde{y} \in \mathcal{L}_Y$ with $\tilde{y} \neq \hat{\hat{y}}_0$ and $F \subset A$ with $\nu(F) > 0$ such that $\tilde{y}(a) \in \mu_a(\hat{y}_0)$ for ν-a.e. $a \in A$ and $p(\hat{y}_0) \cdot \tilde{y}(a) > p(\hat{y}_0) \cdot \hat{\hat{y}}_0(a)$ for ν-a.e. $a \in F$. We thus have

$$\int_F p(\hat{y}_0) \cdot \tilde{y}\, d\nu > \int_F p(\hat{y}_0) \cdot \hat{\hat{y}}_0\, d\nu\,.$$

It follows that

$$p(\hat{y}_0) \cdot \int_F \left(\tilde{y} - \hat{\hat{y}}_0\right) d\nu > 0\,. \tag{11}$$

But for all $\overline{y} \in \mathcal{L}_Y$ with $\overline{y}(a) \in \mu_a(\hat{y}_0)$ for ν-a.e. $a \in F$,

$$p(\hat{y}_0) \cdot \int_F (\tilde{y} - \overline{y}) d\nu = 0\,.$$

Now, $\hat{\hat{y}}_n \longrightarrow \hat{\hat{y}}_0$ in the τ_w^ℓ-topology implies $\lim\limits_n \int_F \hat{\hat{y}}_n d\nu = \int_F \hat{\hat{y}}_0 d\nu$. As each $\hat{\hat{y}}_n$ for $n \in \mathbb{N}$ is bounded by the same integrable function φ introduced in the proof of Lemma 4.4 , by using Proposition 4.1 of Aumann [4, p.7], we get

$$\lim_n \int_F \{\hat{\hat{y}}_n\} d\nu \subset \int_F \limsup_n \{\hat{\hat{y}}(a)\} d\nu$$

where

$$\int_F \limsup_n \{\hat{\hat{y}}(a)\} d\nu$$
$$= \left\{ \int_F \hat{\hat{y}} d\nu \;\middle|\; \hat{\hat{y}}(a) \text{ is a limit point of } \{\hat{\hat{y}}_n(a)\}_{n\in\mathbb{N}} \text{ for } \nu\text{-a.e. } a \in F \right\}.$$

But each limit point of the sequence belongs to $\mu_a(\hat{y}_0)$ except for the set of ν-measure zero. Hence, we must have

$$\begin{aligned} \int_F \tilde{y} d\nu - \lim_n \int_F \hat{\hat{y}}_n d\nu &= \int_F \tilde{y} d\nu - \int_F \hat{\hat{y}}_0 d\nu \\ &= \int_F (\tilde{y} - \hat{\hat{y}}_0) d\nu = 0\,, \end{aligned}$$

because $\hat{\hat{y}}_0 \in \mu_0(\hat{y}_0)$ for ν-a.e. $a \in F$ by the above argument. One thus obtains

$$p(\hat{y}_0)\int_F (\tilde{y} - \hat{\hat{y}}_0) d\nu = 0$$

which contradicts (11). Therefore, $G(\mu)$ must be closed.

□

Proof of the Theorem $L_1(A, \mathcal{A}, \nu)$ endowed with the weak topology is a locally convex linear topological space. Therefore, so is $L_1^\ell(A, \mathcal{A}, \nu)$ endowed with the τ_w^ℓ-toplology. Hence, by Lemma 4.1 through Lemma 4.5, we can apply Ky Fan's Fixed Point Theorem (K. Fan [8, p.122]) to obtain $\hat{y}^* \in \mu(\hat{y}^*)$. It means

$$\hat{y}^*(a) \in \mu_a(\hat{y}^*) \text{ for } \nu\text{-a.e. } a \in A,$$

that is,

$$p(\hat{y}^*) \cdot \hat{y}^*(a) \geq p(\hat{y}^*) \cdot y_a$$

for any $y_a \in Y(a)$, for ν-a.e. $a \in A$. We conclude that $(\hat{y}^*, p(\hat{y}*))$ is an equilibrium point of the production atomless economy $\mathcal{E}_{pro} = [(A, \mathcal{A}, \nu), p, Y]$.

□

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