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Abstract

A *Wicksellian transfer game* is a game composed of three types of players having heterogeneous tastes and heterogeneously endowed commodities among them. In the tradition of economic theory, such a game situation has been considered to represent typical market conditions when traders face transactions among them. In this paper we are interested in finding out the nature of these transfers of commodities resulting from a Wicksellian game.

1 Introduction

A *Wicksellian transfer game* is a game composed of three types of players having heterogeneous tastes and heterogeneously endowed commodities among them. In the tradition of economic theory, such a game situation has been considered to represent typical market conditions when traders face transactions among them. Jevons [5] used the phrase of *"double coincidence of wants"* to emphasize that the "difficulty in barter is to find two persons whose disposable possessions mutually suit each other's wants. There may be many people wanting, and many possessing those things wanted; but to allow of an act of barter there must be a double coincidence, which will rarely happen."

Wickesell in his book [8] described a triangle of trades, which has come to be known as a *Wicksellian triangle*, where each of three traders faces the absence of double coincidence of wants and between any two of the traders one of them has a desire for the commodity

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the different trader possesses, so that a bilateral exchange do not take place between any two of the traders. Wickesell used the triangle to explain how well money facilitates trades without double coincidence of wants.

Now, the absence of double coincidence of wants induces incentives for trading commodities through a third party in order to obtain what one needs or wants. Thus, a Wicksellian triangle among players endogenously creates a circulation or a transfer of commodities, among players, that each player owns. In this paper we examine the nature of these transfers of commodities resulting from a Wicksellian game.

The purpose of the paper is to clarify conditions for a solution of the game to satisfy without specifying an institutional framework under which transfers of commodities take place. As is explained in the following section, we are concerned in this paper with the solution concept of the core of the game. Although the Nash equilibrium of a Wicksellian transfer game can be defined, there is a unique Nash equilibrium that results in no transfers nor trading of commodities among the players as a consequence of absence of binding institutional constraints on transfers.

This paper is organized as follows: In Section 2, we describe a basic Wicksellian transfer game, where three players forming a Wicksellian triangle are classified as three types. Subsequently, the set of all core transfers and a unique competitive equilibrium solution are clarified as the benchmark of our analysis. In Section 3, the basic Wicksellian transfer game is extended to include an additional player of a particular type, say, type 2 player. It creates one other Wiksellian triangle to enlarge opportunity for transfers of commodities. One expects that an increase in the number of type 2 players will induce a qualitative change in possible distribution of commodities among players. In the basic transfer game all the players are both, so to speak, 'monopsony' and 'monopoly' vis-à-vis other players. However, in an extended transfer game, type 2 players face a competition within players of the same type, whereas type 1 and type 3 players continue to enjoy their power of 'monopsony' or that of 'monopoly' as in the basic transfer game. By comparison of its set of core transfers with that of the benchmark case, we shall see how unequal distribution of powers is generated.

Finally, in Section 4 a Wicksellian transfer game with competing players of a same type presented in Section 3, is extended to a general case that n players of a same type participate in Wicksellian triangles. We show how the increase of a type of players give an impact on the other two players.

2 A Wicksellian Transfer Game

2.1 A Basic Transfer Game without Competing Players of a Same Type

There are three commodities/assets, and their typical non-negative combination is expressed by a vector $c = (c_1, c_2, c_3) \in \mathbb{R}^3_+$. In a basic transfer game of this paper there are three players or trading agents, and the set of all these players is denoted by $\mathbb{N} = \{1, 2, 3\}$. Each $i \in \mathbb{N}$ of these players has a specific utility function whose value for a consumption vector $c = (c_1, c_2, c_3) \in \mathbb{R}^3_+$ is given as below:

$$u_1(c) = c_1 + 2c_3$$
$$u_2(c) = c_2 + 2c_1$$
$$u_3(c) = c_3 + 2c_2$$

The initial endowments of each player $i \in \mathbb{N}$ is denoted by e_i and given by

$$e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1).$$

This paper is particularly concerned with specific transfers of commodities/assets among players in $\mathbb{N}$. Thus, a vector $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3_+$ is called a *transfer* when it satisfies $0 \leq \tau_i \leq 1$. As shown in Figure 1 a transfer τ is interpreted to specify that player 1 sends τ_1 amounts of its endowments to player 2, player 2 sends τ_2 amounts of its endowments to player 3, and player 3 sends τ_3 amounts of its endowments to player 1.

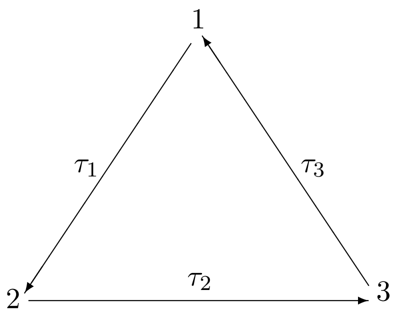

Figure 1: A Transfer $\tau = (\tau_1, \tau_2, \tau_3)$ among Players in a Basic Transfer Model.

When a transfer $\tau = (\tau_1, \tau_2, \tau_3)$ in $\mathbb{N}$ is executed, the resulting consumption vector $c = (c_1, c_2, c_3) \in \mathbb{R}^3_+$ of each player will be given as follows:

$$\text{Player 1: } c_1 = 1 - \tau_1,\ c_2 = 0,\ c_3 = \tau_3$$
$$\text{Player 2: } c_1 = \tau_1,\ c_2 = 1 - \tau_2,\ c_3 = 0$$
$$\text{Player 3: } c_1 = 0,\ c_2 = \tau_2,\ c_3 = 1 - \tau_3$$

An interpretation of a transfer $\tau = (\tau_1, \tau_2, \tau_3)$ might go as follows:

1. Player 1, in planning to obtain τ_3 amounts of the commodity/assets held by player 3, requests player 2 to send τ_2 amounts of its endowments to player 3.

2. Since player 2 prefers player 1's endowments to its own, player 1 effects a transfer of its own endowments by the amount of τ_1, which is enough to motivate player 2 to accept player 1's request.

3. In order for player 2 to be induced to make a transfer of τ_2 amounts to player 3, the amounts τ_1 that player 2 receives from player 1, must be in the range of

$$\frac{1}{2}\tau_2 \leq \tau_1 \leq 1.$$

.

4. In turn, for the amounts τ_1 of the transfer to be justified for player 1, τ_3 must be in the range of

$$0 < \tau_1 \leq 2\tau_3.$$

5. Unless player 3 receives from player 2 the amounts τ_2 satisfying

$$\tau_2 \geq \frac{1}{2}\tau_3,$$

player 3 will not be induced to effect τ_3 amounts of transfer to player 1.

It follows that we have

$$\frac{1}{2}\tau_3 \leq \tau_2 \leq 1.$$

2.2 A Competitive Solution Concept in the Wicksellian Transfer Game

In the basic transfer game here,[1)] each of the players is of the different type, and in this sense competition among them is limitted. Nevertheless, we would like to describe the possible results of market competition among players through the Edgeworthian competition using the game-theoretic concept of the core.[2)] In the standard Walrasian general equilibrium model the market competition among economic agents is described by their price-taking behaviors despite the fact that the price-taking behaviors are hard to be justified when the number of trading agents is small.

Starting from a basic Wicksellian transfer game we will see how an increase in the number of competing players of a same type brings about a reduction in rents gained on the part of various players. In particular, we are interested in how the power of earning rents are distributed among players. Since the transactions of players are explicitly described in this paper by a transfer among them, the core will be defined as a set of transfers among players in contrast to the case of a Walrasian exchange economy in which it is defined as a set of consumption allocations.

[1)] In Fujiki, Green and Yamazaki (2008) a transfer model with asymmetric information is presented to discuss the incentive efficiency of risk-sharing using the core concept.

[2)] A basic model of transfer in this paper may be analyzed as a strategic form game, in which case there is a unique Nash equilibrium transfer $\tau = (\tau_1, \tau_2, \tau_3) = (0, 0, 0)$. Since no one transfers its own endowments at the unique Nash equilibrium, our intended analysis of a market competition among players does not appear to be acheived by an appeal to the Nash equilibrium concept.

Given a transfer $\tau = (\tau_1, \tau_2, \tau_3)$ and $S \subset \mathbb{N}, S \neq \emptyset$, we define the *consumption allocation* $c(\tau) = (c^i(\tau))_{i \in S}$ *among* S *induced by the transfer* τ by[3)]

$$c^i(\tau) = \big(c^i_1(\tau), c^i_2(\tau), c^i_3(\tau)\big) \in \mathbb{R}^3_+, \ i \in S, \text{ where}$$
$$c^i_j(\tau) = \begin{cases} 1 - \tau_i, & \text{if } j = i, \\ 0, & \text{if } j = i+1, \\ \tau_{i-1}, & \text{if } j = i-1, \end{cases}$$
$$\text{for } j = 1, 2, 3.$$

For $S \subset \mathbb{N}, S \neq \emptyset$, a transfer $\tau = (\tau_1, \tau_2, \tau_3)$ is *feasible for* S if

$$(\forall i \in \mathbb{N}) \ i \notin S \text{ or } (i+1) \notin S \text{ implies } \ \tau_i = 0.$$

S is said to *improve upon* τ if there exists a transfer $\mu = (\mu_1, \mu_2, \mu_3) \in [0,1]^3$ which is feasible for S and the consumption allocation $c(\mu)$ among S induced by μ satisfies

$$(\forall i \in S) \ u_i(c^i(\mu)) \geq u_i(c^i(\tau)), \text{ and}$$
$$(\exists i \in S) \ u_i(c^i(\mu)) > u_i(c^i(\tau)).$$

A transfer τ is *Pareto Optimal* (PO) if $\mathbb{N}$ itself cannot improve upon τ. It is *individually rational* (IR) if $S = \{i\}$ cannot improve upon τ for all $i \in \mathbb{N}$. τ is a *core transfer* if there are no $S \subset \mathbb{N}, S \neq \emptyset$, that can improve upon it. The *core* is the set of all core transfers. [4)]

2.3 The Core of the Basic Transfer Game

In the basic transfer model with the set $\mathbb{N} = \{1, 2, 3\}$ of players it is not possible for a set $S \subset \mathbb{N}$ consisting of two players to improve upon a transfer τ in $\mathbb{N}$. Indeed, take a set S consisting of players i and $i+1$. Their utilities of the consumption allocation induced by the transfer τ is given by

$$u_i(c^i(\tau)) = (1 - \tau_i) + 2\tau_{i-1}$$
$$u_{i+1}(c^{i+1}(\tau)) = (1 - \tau_{i+1}) + 2\tau_i.$$

3) Throughout this paper we set $i + 1 = 1$ if $i = 3$, and $i - 1 = 3$ if $i = 1$ for our notational convenience. Note also that we have $i + 1 = i - 2$ and $i - 1 = i + 2$.

4) When one views the transfer model as an exchange economy, one might suspect whether the core as a set of transfers in $\mathbb{N}$ is equivalent to the core as a set of consumption allocations as a transfer in $\mathbb{N}$ reflects a micro-market structure of trading. One can check tthe following fact:

Fact: If a transfer $\tau = (\tau_1, \tau_2, \tau_3) \in [0,1]^3$ in $\mathbb{N}$ is in the core, then the consumption allocation $c(\tau) = (c^i(\tau))_{i \in \mathbb{N}}$ induced by the transfer τ in $\mathbb{N}$ is a core allocation. Conversely, if an allocation $f : \mathbb{N} \to \mathbb{R}^3_+$ is in the core of the exchange economy, then there exists a core transfer $\tau = (\tau_1, \tau_2, \tau_3) \in [0,1]^3$ in $\mathbb{N}$ such that the consumption allocation $c(\tau) = (c^i(\tau))_{i \in \mathbb{N}}$ induced by the transfer τ satisfies $(\forall i \in \mathbb{N}) \ f(i) = c^i(\tau)$.

Thus, the value of τ_i must be reduced to increase the utility of i whereas its value need to be increased to increase the utility of $i+1$. It follows that, with regard to the utilities of players i and $i+1$, it would not be possible to increase the utility of one without reducing that of the other. Hence, in case of the basic transfer game, the condition for a transfer τ in $\mathbb{N}$ to be a core transfer is simply that τ satisfies the both individual rationality (IR) and the Pareto optimality (PO).

We first turn to the condition for a transfer τ to be Pareto optimal. Suppose that for all $i \in \mathbb{N}$ we had $\tau_i < 1$. Without loss of generality, assume $\tau_1 = \max\{\tau_i | i = 1, 2, 3\}$. Setting $\varepsilon = 1 - \tau_1$, define a new transfer μ as follows:

$$\begin{aligned}
\mu_1 &= \tau_1 + \varepsilon = 1 \\
\mu_2 &= \tau_2 + \frac{\varepsilon}{2} \\
\mu_3 &= \tau_3 + \frac{\varepsilon}{2}
\end{aligned}$$

By the way μ is defined, one can check that it is indeed a transfer in $\mathbb{N}$. The consumption allocation $c(\mu) = (c^i(\mu))_{i \in \mathbb{N}}$ then gives rise to the utilities as below:

$$\begin{aligned}
u_1(c^1(\mu)) &= (1 - \mu_1) + 2\mu_3 \\
&= (1 - \tau_1 - \varepsilon) + 2\left(\tau_3 + \frac{\varepsilon}{2}\right) \\
&= u_1(c^1(\tau)) \\
u_2(c^2(\mu)) &= (1 - \mu_2) + 2\mu_1 \\
&= \left(1 - \tau_2 - \frac{\varepsilon}{2}\right) + 2\left(\tau_1 + \varepsilon\right) \\
&= u_2(c^2(\tau)) + \frac{3}{2}\varepsilon \\
&> u_2(c^2(\tau)) \\
u_3(c^3(\mu)) &= (1 - \mu_3) + 2\mu_2 \\
&= \left(1 - \tau_3 - \frac{\varepsilon}{2}\right) + 2\left(\tau_2 + \frac{\varepsilon}{2}\right) \\
&= u_3(c^3(\tau)) + \frac{\varepsilon}{2} \\
&> u_3(c^3(\tau)).
\end{aligned}$$

This shows that $\mathbb{N}$ can improve upon τ, contradicting to the fact τ is PO. Therefore, if τ is PO, then one must have

$$(\exists i \in \mathbb{N})\tau_i = 1.$$

Conversely, suppose for some $k \in \mathbb{N}$ we have $\tau_k = 1$. Then, for the consumption allocation $c(\tau) = (c^i(\tau))_{i\in\mathbb{N}}$ induced by τ, the utilities of players are given by

$$
\begin{aligned}
u_k(c^k(\tau)) &= 2\tau_{k-1} \\
u_i(c^i(\tau)) &= 1 - \tau_i + 2\tau_{i-1} \\
&= \begin{cases} 3 - \tau_{k+1}, & \text{if } i = k+1\ , \\ 1 - \tau_{k-1} + 2\tau_{k+1}, & \text{if } i = k-1\ . \end{cases}
\end{aligned} \tag{1}
$$

Assume that $\mathbb{N}$ improved upon the transfer τ by another transfer μ; then, we would have

$$1 - \mu_k + 2\mu_{k-1} \geq 2\tau_{k-1} \tag{2}$$

$$1 - \mu_{k+1} + 2\mu_k \geq 3 - \tau_{k+1} \tag{3}$$

$$1 - \mu_{k-1} + 2\mu_{k+1} \geq 1 - \tau_{k-1} + 2\tau_{k+1}. \tag{4}$$

with at least one inequality holding strictly. It follows from (3) and (4) with regards to players $k+1$ and $k-1$

$$\mu_{k+1} - \tau_{k+1} \leq 2(\mu_k - 1) \tag{5}$$

$$\mu_{k-1} - \tau_{k-1} \leq 2(\mu_{k+1} - \tau_{k+1}) \tag{6}$$

Noting that $0 \leq \mu_k \leq 1$, by comparison of the utilities of player k obtained by the consumption allocation induced by transfers τ and μ, it follows from (5) and (6) that we have a series of inequalities below:

$$
\begin{aligned}
0 \ \leq \ u_k(c^k(\mu)) - u_k(c^k(\tau)) &= (1 - \mu_k) + 2(\mu_{k-1} - \tau_{k-1}) \\
&\leq (1 - \mu_k) + 4(\mu_{k+1} - \tau_{k+1}) \\
&\leq (1 - \mu_k) + 8(\mu_k - 1) \\
&= 7(\mu_k - 1) \leq 0,
\end{aligned} \tag{7}
$$

where at least one of the inequalities must hold strictly, resulting in a contradiction. Therefore, if $\tau_k = 1$, then $\mathbb{N}$ can not improve upon the transfer τ.

We thus obtain the following:

Proposition 2.1 *A transfer τ in $\mathbb{N} = \{1, 2, 3\}$ is Pareto optimal if and only if we hve*

$$(\exists i \in \mathbb{N})\tau_i = 1.$$

We now confirm the condition for the individual rationality. A transfer τ satisfies the individual rationality (IR) if and only if we have for all $i \in \mathbb{N}$ and for the consumption allocation $c(\tau)$ induced by τ

$$
\begin{aligned}
u_i(c^i(\tau)) &= 1 - \tau_i + 2\tau_{i-1} \\
&\geq u_i(e_i) = 1
\end{aligned}
$$

Therefore, the condition for the individual rationality is given by

$$(\forall i \in \mathbb{N})\ \tau_i \leq 2\tau_{i-1} \tag{8}$$

Since the condition for a transfer to be a core transfer in the basic transfer game is that it satisfies both PO and IR, we summarize the above arguments as follows:

Proposition 2.2 *A transfer τ in the basic Wicksellian transfer game is a core transfer if and only if for some $i \in \mathbb{N}$ we have*

$$\tau_i = 1,\ \tau_{i-1} \in \left[\frac{1}{2}, 1\right],\ \tau_{i+1} \in \left[\frac{\tau_{i-1}}{2}, 1\right]$$

Essentially there are no competing players of the same type in the basic transfer game taken up in this section. For this reason the solution concept of the core does not circumscribe resulting transfers tightly enough. Consider, for example, the case of $\tau_3 = 1$ in the above proposition which describes circumstances where player 3 makes the maximum amounts of transfer of its endowments to player 1. The core does not give a definitive answer as to who between players 1 and 2 receives higher benefit by a transfer in the core. Depending upon a core transfer player 1 or player 2 gets higher rents than the other.

Nonetheless, one could expect that, by an emergence of competing players of the same type who participate in a transfer of commodities/assets, the rents of an agent facing competitors decline as the number of its competitors increases. Hence, in section 3, we will introduce in the basic transfer game a competitive environment by adding a competitor of a player, and see as a consequence what the core predicts as a result of competition in trading.

2.4 Walrasian Analysis in the Basic Transfer Game

Before proceeding to introducing a competitive environment among players of the same type in an explicit way, we now briefly turn to the Walrasian analysis of the basic Wicksellian transfer game, viewed as a general equilibrium market transaction model.

Let us first set the notation for some standard concepts. For each player $i \in \mathbb{N}$, denote the budget set (the budget surface) and the demand set at a price vector $p \in \mathbb{R}^3_+$, respectively, by

$$\beta^i(p) = \{z \in \mathbb{R}^3_+ | p \cdot z \leq p \cdot e_i\} \quad (\bar{\beta}^i(p) = \{z \in \mathbb{R}^3_+ | p \cdot z = p \cdot e_i\}),$$
$$\varphi^i(p) = \{x \in \beta^i(p) | \beta^i(p) \cap \{z \in \mathbb{R}^3_+ | u^i(z) > u^i(x)\} = \emptyset\}.$$

In general $\varphi^i(p)$ is set-valued. By an abuse of notation we might also express coordinatewise a (demand) selection from $\varphi^i(p)$ by

$$\varphi^i(p) = \left(\varphi^i_1(p), \varphi^i_2(p), \varphi^i_3(p)\right)$$
$$\left(= \left(\varphi^i_j(p)\right)_j\right) \in \mathbb{R}^3_+.$$

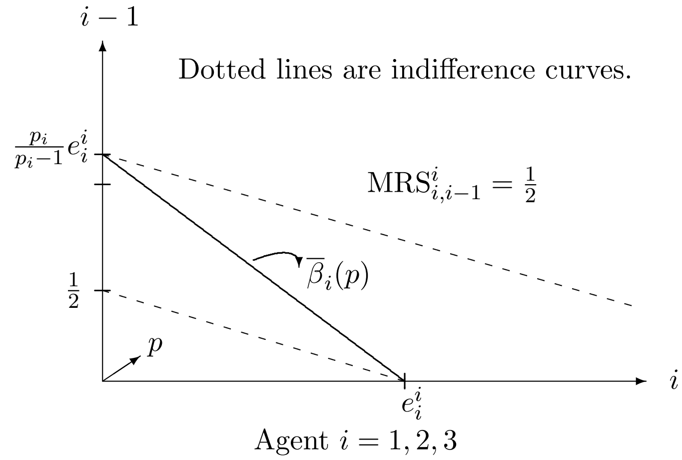

Figure 2: The Budget Set and Indifference Curves

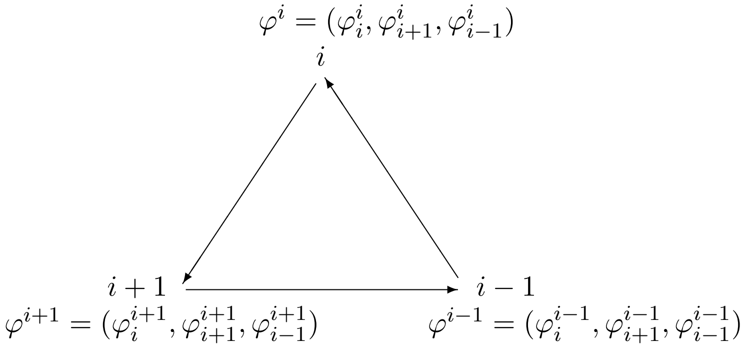

Figure 3: The Consumption Demand of Agents

Now, $f : \mathbb{N} \to \mathbb{R}^3_+$ satisfying

$$\sum_{i\in\mathbb{N}} f(i) = \sum_{i\in\mathbb{N}} e_i \tag{9}$$

is an allocation. It is a *Walras allocation* if there exists a price vector $p \in \mathbb{R}^\ell_{++}$ such that one has

$$(\forall i \in \mathbb{N}) f(i) \in \varphi^i(p).$$

A price vector p associated with a Walras allocation f is called an *equilibrium price vector* or *equilibrium prices*, and the pair (p, f), a *Walras equilibrium.*

We now search for equilibrium prices and Walras allocations. If $p_i = 0$ for some $i = 1, 2, 3$, then for any sequence $(p^n)_n$ of price vectors such that $(p^n)_n \longrightarrow p$ we have $\varphi^{i+1}_i(p^n) \longrightarrow \infty$, showing that we only need to check price vectors in $\mathbb{R}^3_{++}$.

For the purpose of our argument, let $\Delta_+ := \{p' \in \mathbb{R}^3_{++} | \sum_{k=1}^3 p'_k = 1\}$. By examining the budget set and indifference curves of agent $i = 1, 2, 3$, in Figure 2, one can check that the demands $\varphi^i(p) = (\varphi^i_i(p), \varphi^i_{i+1}(p), \varphi^i_{i-1}(p))$ of each player $i = 1, 2, 3$, are given as follows:

$$\varphi^i_i(p) = \begin{cases} 0 & \text{for } \frac{p_i}{p_{i-1}} > \frac{1}{2} \\ \overline{q} & \text{for } \frac{p_i}{p_{i-1}} = \frac{1}{2} \\ e^i_i & \text{for } 0 < \frac{p_i}{p_{i-1}} < \frac{1}{2} \end{cases} \tag{10}$$

$$\varphi^i_{i+1}(p) = 0 \tag{11}$$

$$\varphi^i_{i-1}(p) = \begin{cases} \dfrac{p_i}{p_{i-1}} e^i_i & \text{for } \frac{p_i}{p_{i-1}} > \frac{1}{2} \\ \dfrac{1}{2}(e^i_i - \overline{q}) & \text{for } \frac{p_i}{p_{i-1}} = \frac{1}{2} \\ 0 & \text{for } 0 < \frac{p_i}{p_{i-1}} < \frac{1}{2} \end{cases} \tag{12}$$

with $\overline{q} \in [0,1]$ and $p \in \Delta_+$. Since one has

$$\prod_{i=1,2,3} \frac{p_i}{p_{i-1}} = 1,$$

there must be at least one i for which $\dfrac{p_i}{p_{i-1}} > \dfrac{1}{2}$. Let k be such an i so that we have $\dfrac{p_k}{p_{k-1}} > \dfrac{1}{2}$, which implies, by (10) and (12), $\varphi^k_k(p) = 0$ and $\varphi^k_{k-1}(p) = \dfrac{p_k}{p_{k-1}} e^k_k$ respectively.

Assume p is an equilibrium price vector; then, $(\varphi^i(p))_{i=1,2,3}$ must be an allocation. As $\varphi^{k-1}_k(p) = 0$ from (11), one must have from (12) that

$$\sum_{i=1,2,3} \varphi^i_k(p) = \varphi^{k+1}_k(p) = \frac{p_{k+1}}{p_k}\left(e^{k+1}_{k+1} - \varphi^{k+1}_{k+1}(p)\right) = e^k_k \; . \tag{13}$$

It thus follows that

$$\frac{p_{k+1}}{p_k} = \frac{e^k_k}{e^{k+1}_{k+1} - \varphi^{k+1}_{k+1}(p)} \geq \frac{e^k_k}{e^{k+1}_{k+1}} \; , \tag{14}$$

which implies $\dfrac{p_{k+1}}{p_k} \geq 1$. Then, by (10), $\varphi^{k+1}_{k+1}(p) = 0$, and (14) implies

$$\frac{p_{k+1}}{p_k} = 1 \; . \tag{15}$$

Similarly, since we have from (11)

$$\begin{aligned} \sum_{i=1,2,3} \varphi^i_{k-1}(p) &= \varphi^k_{k-1}(p) + \varphi^{k-1}_{k-1}(p) \\ &= \frac{p_k}{p_{k-1}} e^k_k + \varphi^{k-1}_{k-1}(p) \\ &= e^{k-1}_{k-1} \; , \end{aligned} \tag{16}$$

$\varphi^{k-1}_{k-1}(p) = e^{k-1}_{k-1}$ for $0 < \dfrac{p_{k-1}}{p_{k+1}} < \dfrac{1}{2}$ implies that we must have $\dfrac{p_{k-1}}{p_{k+1}} \geq \dfrac{1}{2}$. As we have

$$\begin{aligned} \sum_{i=1,2,3} \varphi^i_{k+1}(p) &= \varphi^{k+1}_{k+1}(p) + \varphi^{k-1}_{k+1}(p) = \varphi^{k-1}_{k+1}(p) \\ &= \frac{p_{k-1}}{p_{k+1}}\left(e^{k-1}_{k-1} - \varphi^{k-1}_{k-1}(p)\right) = e^{k+1}_{k+1} \; , \end{aligned} \tag{17}$$

by the same arguments leading to (15) one obtains $\dfrac{p_{k-1}}{p_{k+1}} = 1$. Hence, we also have $\dfrac{p_{k-1}}{p_k} = 1$. Therefore, for $p \in \Delta_+$ to be an equilibrium price vector, it must satisfy

$$p_1 = p_2 = p_3 = \frac{1}{3} \,. \tag{18}$$

Conversely, if a price vector p satisfies (18), then the corresponding $(\varphi^i(p))_{i=1,2,3}$ generates a Walras allocation. Hence, p satisfying (18) is a unique equilibrium price in Δ_+.

A Walras allocation $f : \mathbb{N} \to \mathbb{R}^3_+$ associated with this unique equilibrium price vector p is also uniquely determined as [5)]

$$\begin{aligned} f(1) &= \varphi^1(p) = (0, 0, 1) \,, \\ f(2) &= \varphi^2(p) = (1, 0, 0) \,, \\ f(3) &= \varphi^3(p) = (0, 1, 0) \,. \end{aligned} \tag{19}$$

One can also see that there is a unique transfer among $\mathbb{N}$ that gives rise to the Walars allocation above. In fact, given $(\varphi^i)_{i\in\mathbb{N}}$, the demand vectors it specifies might be enacted by a transfer in $\mathbb{N}$ of the basic transfer game. Let a transfer $\tau^{\varphi(p)}$ among $\mathbb{N}$ defined by

$$(\forall i \in \mathbb{N})\, \tau_i^{\varphi(p)} = 1 - \varphi_i^i(p) \tag{20}$$

is called the *φ-transfer*. When prices upon which players' demands depend are clearly understood, a φ-transfer is often written simply as τ^φ or $(\tau_i^\varphi)_{i\in\mathbb{N}}$ without any reference to a price vector p. By the way the φ-transfer among $\mathbb{N}$ is defined, it leads each player's consumption to its demand vector. In other words, one can confirm that we have

$$(\forall i \in \mathbb{N})\, \varphi^i(p) = c^i\left(\tau^\varphi\right), \tag{21}$$

that is, the consumption allocation induced by the φ-transfer gives each agent its demand vector. We note that the the φ-transfer defined by (20) satisfies

$$(\forall i \in \mathbb{N})\, \varphi_{i-1}^i = \tau_{i-1}^\varphi = 1 - \varphi_{i-1}^{i-1}. \tag{22}$$

The consumption allocation induced by the φ-transfer at the equilibrium price vector gives the Walras allocation

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5) In Section 3 the basic model of transfer is extended to allow for a competing agent of type 2 agent. One can check that the p satisfying (18) is still the unique equilibrium price in Δ_+ for the extended transfer model of Section 3. Indeed, for the transfer model of Section 3 with $\mathbb{N} = \{1, 2_1, 2_2, 3\}$, in case $i \in \mathbb{N}$ being $i = 2_j, j = 1, 2$, $\varphi^i(p) = (\varphi_i^i(p), \varphi_{i+1}^i(p), \varphi_{i-1}^i(p)) = (\varphi_2^{2_j}(p), \varphi_3^{2_j}(p), \varphi_1^{2_j}(p))$ for $i = 2_j$ and $e_2^{2_j} = \frac{1}{2}$ in reading from (10) to (12). From (13) to (17), the endowment vector e_k^k, when it refers to type 2 agents, e_2^2 should be understood to represent the sum of their endowments so that $e_2^2 = \sum_{j=1,2} e_2^{2_j} = 1$. Then, one can see that the price vector p satisfying (18) is the unique equilibrium price, and the associated Walras allocation is given as in (18) except for the type 2 agent $2_j, j = 1, 2$, which is given by $f(2_j) = \varphi^{2_j}(p) = (1/2, 0, 0)$. Similarly, it is not difficult to see that even when the number of type 2 agents only is increased to n, the price vector p satisfying (18) is still the unique equilibrium price, and the associated Walras allocation is given as in (18) except for the type 2 agent $2_j, j = 1, \ldots, n$, which is given by $f(2_j) = \varphi^{2_j}(p) = (1/n, 0, 0)$.

Remark 1 As shown in the Walrasian analysis of the basic transfer game, the price-taking behavior of players in market transactions, pinpoints the result of competition in this case despite the fact that there is hardly any competition among players. All the possible rents that might be gained by some of the players are entirely eliminated. In a sense it is achieved by the fact that all the players participating in market transactions have given up their power to "negotiate" with the markets and among themselves by taking market prices as given. It is in stark contrast with the core of the basic transfer game in which, as shown in Proposition 2.2, there are so much rooms for possible gains as rents. In our analysis below how the emergence of competitors among the players narrows down possible rents among players.

3 A Wicksellian Transfer Game with Competing Players of a Same Type

3.1 Adding a Competing Player to the Basic Transfer Game

Scenario of Transactions In the basic transfer game of the previous section an initial analysis has been carried out within the framework of a transfer of commodities/assets in the case where there are three players each of whom wishes to obtain a commodity/asset held by other agent in an indirect exchange for a commodity/asset that itself is endowed with as means of its payment. From player $1'$s point of view, the structure of these transfers might be understood in the following way: Player 1 wishes to procure some amounts of the commodity/asset held by player 3. Player 3, however, does not particularly care for getting the commodity/asset held by the player 1. Hence, knowing that player 2 holds the commodity/asset which player 3 wishes to acquire and that at the same time likes to receive some amounts of what the player 1 has as its endowments, the player 1 asks the player 2 to transfer a part of its endowments to the player 3. In return the player 1 sends it some amounts of its endowments as its payments for its request of the transfer.

In this section we are going to extend this basic framework for transfers among players in a most simple way introducing a competitive factor with respect to transactions among players. Within the basic framework, when player 1 requests player 2 to transfer a part of its endowments to player 3 in order to obtain some amounts of player $3'$s endowments, this "transfer or payment service" could be provided by only one player in the entire economy; consequently, player 2 holds, so to speak, a monopolistic position and conceivably acquires "rents" in such a role. Accordingly, to remove a monopolistic position of a player in the basic transfer game, a competitive environment shall be substantiated by adding to the game one other player that exactly fulfills the same role as player 2. These two players are named as players 2_1 and 2_2 to signify they have essentially the same characteristics.

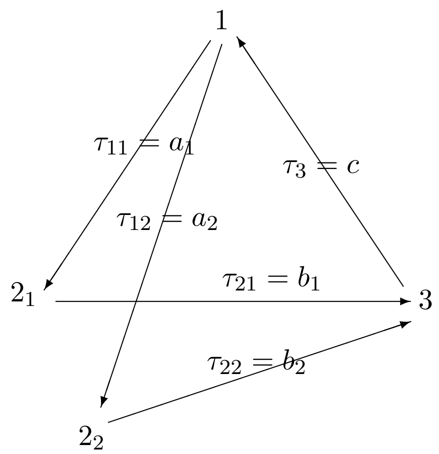

Figure 4: A Transfer $\tau = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \tau_3) = (a_1, a_2, b_1, b_2, c)$ in Competing Wicksellian Triangles

In other words, players 2_1 and 2_2 are a "copy" or a "replica" of the other and have the same utility function and the initial endowments vector.

If one denotes the Wicksellian triangle in the basic transfer game by $\Delta 123$, in this section a system of transfers of commodities/assets with a framework having two Wicksellian triangles given by $\Delta 12_1 3$ and $\Delta 12_2 3$ is examined. Thus, the set of all players is given by

$$\mathbb{N} = \{1, 2_1, 2_2, 3\}.$$

The utility functions, u_1, u_3, and the initial endowments vectors, e_1, e_3, of players 1 and 3 are same as the ones in the basic transfer game of the previous section. As to players 2_1 and 2_2, their utility functions, u_{21}, u_{22}, are same as that of the player $2'$s utility function u_2, but their initial endowments vectors, e_{21}, e_{22}, are given so as to make the total endowments as a whole remain unchanged; thus,

$$u_{21} = u_{22} = u_2, e_{21} = e_{22} = \frac{1}{2} e_2$$

where the suffix $2j$ of utility functions or initial endowments vectors indicates an associated respective players.

A transfer τ among $\mathbb{N} = \{1, 2_1, 2_2, 3\}$ is now denoted by

$$\tau = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \tau_3)$$

where i in the suffix of τ_{ij} shows the *sender* of the transfer. For $i = 1$, j shows the *receiver* 2_j, and for $i = 2$, j shows the *sender* 2_j. The Figure 4 shows a transfer composed of two Wicksallian triangles. In the basic transfer game of the preceding section, simply $\tau \in [0, 1]^3$ in general is called a transfer. In this section, however, $\tau = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \tau_3) \in [0, 1]^5$ is called a *transfer* among $\mathbb{N}$ if it satisfies the constraint

$$0 \leq \tau_{2j} \leq \frac{1}{2}, \ j = 1, 2,$$

that is, a transfer must be done within the obvious constraint given by each player's initial endowments.

By a comparison of Figure 1 and Figure 4 one can see that the essential difference in the frameworks between the basic game and its extension here is that players 2_1 and 2_2 are now outright competitors to each other. In line with the scenario of this section player 1 in Figure 1 has a transfer route only through player 2 whereas in Figure 4 player 1 has a choice of routes between one through player 2_1 and the other through player 2_2. In other words player 1 is now in a position to exercise its judgment as to who offers player 1 more favorable terms of transfer.

3.2 The Core of the Transfer Game

A transfer $\tau = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \tau_3) \in [0,1]^5$ among $\mathbb{N}$ is defined to be a *core transfer* exactly as in the previous section.[6)] For a transfer $\tau \in [0,1]^5$ among $\mathbb{N}$, τ_1 and τ_2 are

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[6)] The analysis of this paper will be continued so as to add successively a copy or a replica of a type of players. To avoid the tedious repetition of parallel definition and notation, we would like to define a transfer and a core transfer among $\mathbb{N}$ in a general setup.

Now, letting $J_i, i = 1,2,3$, be the set of all copies of the type i players, set $\mathbb{N} = \{i_j | \, i = 1,2,3, j \in J_i\}$. Denote by $\tau_{ijk} \in [0,1], i = 1,2,3, j \in J_i, k \in J_{i+1}$, the amounts that are transferred by player i_j to player $(i+1)_k$. A *transfer* τ *among* $\mathbb{N}$, written as $\tau = (\tau_{ijk})_{i=1,2,3,j\in J_i,k\in J_{i+1}}$ is such that for each player the amounts that it transfers to other players do not exceed the amounts of its endowments so that

$$(\forall i = 1,2,3)(\forall j \in J_i) \; 0 \leq \sum_{k\in J_{i+1}} \tau_{ijk} \leq \frac{1}{\sharp J_i} \; .$$

The *consumption allocation* $c(\tau)$ *induced by a transfer* τ is a list composed of consumption vectors $c^{ij}(\tau)$ of each player i_j such that

$$c(\tau) = \left(c^{ij}(\tau)\right)_{j\in J_i, i=1,2,3} \; ,$$

$$c^{ij}_\ell(\tau) = \begin{cases} \frac{1}{\sharp J_i} - \sum_{k\in J_{i+1}} \tau_{ijk}, & \text{for } \ell = i \; , \\ 0, & \text{for } \ell = i+1 \; , \\ \sum_{k\in J_{i-1}} \tau_{(i-1)kj}, & \text{for } \ell = i-1 \; . \end{cases}$$

Hence, the utility of each player $i_j, i = 1,2,3, j \in J_i$, gained by the consumption allocation $c(\tau)$ is

$$u_{ij}\left(c^{ij}(\tau)\right) = \left(\frac{1}{\sharp J_i} - \sum_{k\in J_{i+1}} \tau_{ijk}\right) + 2 \sum_{k\in J_{i-1}} \tau_{(i-1)ki_j} .$$

A transfer $\tau = (\tau_{ijk})_{i=1,2,3,j\in J_i,k\in J_{i+1}}$ among $\mathbb{N}$ is *feasible for* $S \subset \mathbb{N}$ if

$$(\forall i_j \in \mathbb{N}) \; i_j \notin S \text{ or } (i+1)_k \notin S \text{ implies } \; \tau_{ijk} = 0.$$

$S \subset \mathbb{N}$ *improves upon* a transfer τ if there exists a transfer $\mu = (\mu_{ijk})_{i=1,2,3,j\in J_i,k\in J_{i+1}}$ (among $\mathbb{N}$) that is feasible for S such that the consumption allocation $c(\mu)$ induced by μ satisfies

$$(\forall i_j \in S) \; u_{ij}(c^{ij}(\mu)) \geq u_i(c^{ij}(\tau)) \; ,$$
$$(\exists i_j \in S) \; u_{ij}(c^{ij}(\mu)) > u_{ij}(c^{ij}(\tau)) \; .$$

τ is a *core transfer* if S cannot improve upon τ for any $S \subset \mathbb{N}, S \neq \emptyset$.

defined, for our convenience of expression, to be as follows:

$$\tau_i = \tau_{i1} + \tau_{i2},\ i = 1, 2\ .$$

The condition for τ to be a core transfer in the previous section where $\mathbb{N} = \{1, 2, 3\}$ is that it only satisfies both of the PO and IR conditions. In case of $\mathbb{N} = \{1, 2_1, 2_2, 3\}$ of this section, the condition that neither $\{1, 2_1, 3\}$ nor $\{1, 2_2, 3\}$ can improve upon τ, must be added to the conditions that τ satisfies both of the PO and IR conditions.

Let us first check the IR condition for each player in $\mathbb{N}$. It is given by the following inequalities:

$$\begin{aligned} u_1(c^1(\tau)) &= 1 - \tau_1 + 2\tau_3 \geq 1 \\ u_{2j}(c^{2j}(\tau)) &= \frac{1}{2} - \tau_{2j} + 2\tau_{1j} \geq \frac{1}{2},\ j = 1, 2 \\ u_3(c^3(\tau)) &= 1 - \tau_3 + 2\tau_2 \geq 1 \end{aligned}$$

Thus, the IR condition for each player in $\mathbb{N}$ becomes:

$$\tau_3 \geq \frac{1}{2}\tau_1\ ,\ \tau_{1j} \geq \frac{1}{2}\tau_{2j}\ (j = 1, 2)\ ,\ \tau_2 \geq \frac{1}{2}\tau_3\ . \tag{23}$$

Next, we check the PO condition for a transfer τ among $\mathbb{N}$. Let τ be Pareto optimal. Then, by exactly the same arguments as in the previous section, one can show that one must have

$$(\exists i = 1, 2, 3)\ \tau_i = 1\ .$$

Now, we show that the condition is sufficient for τ to be Pareto optimal. The essence of arguments for its sufficiency is also the same as in the proof of Proposition 2.1 except for the care to be taken to account for plural type 2 players. To avoid a repetition of our arguments, we explicate changes to be made in the proof of sufficiency in Proposition 2.1.

If $\tau_k = 1$ for $k = 1$ or $k = 3$, take the utility level of the player 2 in (1) to be the sum of those of type 2 players, 2_1 and 2_2, that is,

$$u_2(c^2(\mu)) = u_{21}(c^{21}(\mu)) + u_{22}(c^{22}(\mu))\ .$$

Suppose $\mathbb{N}$ improved upon the transfer τ by another transfer μ; then, by setting $\tau_i = \tau_{i1} + \tau_{i2}$ and $\mu_i = \mu_{i1} + \mu_{i2}$ for $i = 1, 2$, in (2) through (7), one sees that a series of inequalities in (7) leads to a contradiction since at least one of them holds with strict inequality.

If $\tau_2 = 1$ so that $\tau_{21} = \tau_{22} = \frac{1}{2}$, this corresponds to the case of $k = 2$ in (1) through (7), in each of which we apply the above changeover of the terms. Then, by setting in (7)

$$\begin{aligned} u_2(c^2(\mu)) &= u_{21}(c^{21}(\mu)) + u_{22}(c^{22}(\mu))\ , \\ u_2(c^2(\tau)) &= u_{21}(c^{21}(\tau)) + u_{22}(c^{22}(\tau))\ , \end{aligned}$$

a series of inequalities in (7) leads to a contradiction.

It follows that there are no transfers $\mu \in [0, 1]^5$ among $\mathbb{N}$ by which $\mathbb{N}$ could improve upon τ. Hence, we obtain the following:

Proposition 3.1 *A transfer* $\tau = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \tau_3) \in [0,1]^5$ *among* $\mathbb{N} = \{1, 2_1, 2_2, 3\}$ *is Pareto optimal if and only if* $\tau_i = 1$ *for some* $i = 1, 2, 3$.

Let us now turn to $S \subset \mathbb{N}$ with three players, and check conditions under which it cannot improve upon the transfer τ. Among $S \subset \mathbb{N}$ with tree players, those containing players 2_1 and 2_2 simultaneously cannot improve upon the transfer τ. Thus, one only needs to check the subsets $\{1, 2_1, 3\}$ and $\{1, 2_2, 3\}$.

Here one might expect that the condition that these $S \subset \mathbb{N}$ cannot improve upon the transfer τ is that, in decomposing the transfer τ into two Wicksell triangles, $\Delta 12_1 3$ and $\Delta 12_2 3$, τ restricted to each $\Delta 12_j 3$, $j = 1, 2$, satisfies the PO condition in the respective subgame; that is, just as the PO condition in the basic Wicksellian transfer game, $\{1, 2_j, 3\}$ cannot improve upon τ if and only if there is $i \in \{1, 2_j, 3\}$ such that $\tau_i = 1$ if $i = 1, 3$, or $\tau_i = 1/2$ if $i = 2j$. However, when $\mathbb{N} = \{1, 2_1, 2_2, 3\}$, players 1 and 3 may avail themselves, so to speak, an escape route $\Delta 12_k 3$, $k \neq j$, for the route $\Delta 12_j 3$. Therefore, the condition that $\{1, 2_j, 3\}$ cannot improve upon τ differs from the PO condition within the Wicksell triangle $\Delta 12_j 3$.

Henceforth, we look for conditions under which the transfer τ among $\mathbb{N} = \{1, 2_1, 2_2, 3\}$ cannot be improved upon by $S_j \equiv \{1, 2_j, 3\}$, $j = 1, 2$. Let us represent the transfer τ by the value of parameters as follows:

$$\tau = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \tau_3) = (a_1, a_2, b_1, b_2, c)$$

(see Figure 4). Since τ is Pareto optimal, by Proposition 3.1, at least one of the following equalities

$$a_1 + a_2 = 1, \ b_1 = b_2 = \frac{1}{2}, \ c = 1 \tag{24}$$

holds. Moreover, by the IR condition (23) the value of these parameters satisfies the inequalities

$$\begin{aligned} a_1 \geq \frac{1}{2} b_1 \ , \ a_2 \geq \frac{1}{2} b_2 \ , \\ b_1 + b_2 \geq \frac{1}{2} c \ , \\ c \geq \frac{1}{2}(a_1 + a_2) \ . \end{aligned} \tag{25}$$

We use the notation a, b to represent the sum of the corresponding a_i or b_i so that

$$a = a_1 + a_2, \ b = b_1 + b_2$$

in this section.

Let us examine the possibility of S_j to improve upon the transfer τ. For this purpose define $\mu = (\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_3) \in [0,1]^5$ by

$$\begin{aligned} \mu_{1i} &= \begin{cases} a_i + \varepsilon_i & \text{if } i = j \ , \\ 0 & \text{if } i \neq j \ , \end{cases} \\ \mu_{2i} &= \begin{cases} b_i + \delta_i & \text{if } i = j \ , \\ 0 & \text{if } i \neq j \ , \end{cases} \\ \mu_3 &= c + \gamma \end{aligned} \tag{26}$$

where $\gamma, \varepsilon_i, \delta_i$ for $i = j$ are real-valued parameters to be determined so as to make μ a transfer among $\mathbb{N}$.

Now, if μ is indeed a transfer among $\mathbb{N}$, it is feasible for S_j. The utility difference for each player in S_j between the consumption allocations induced by μ and τ is given as follows:

$$\begin{aligned} u_1(c^1(\mu)) - u_1(c^1(\tau)) &= (1 - a_j - \varepsilon_j) + 2(c + \gamma) - (1 - a + 2c) \\ &= 2\gamma + (a_i - \varepsilon_j) \\ u_{2j}(c^{2j}(\mu)) - u_{2j}(c^{2j}(\tau)) &= \left(\frac{1}{2} - b_j - \delta_j\right) + 2(a_j + \varepsilon_j) - \left(\frac{1}{2} - b_j + 2a_j\right) \\ &= 2\varepsilon_j - \delta_j \\ u_3(c^3(\mu)) - u_3(c^3(\tau)) &= (1 - c - \gamma) + 2(b_j + \delta_j) - (1 - c + 2b) \\ &= 2(\delta_j - b_i) - \gamma \end{aligned}$$

where $i, j = 1, 2,\ i \neq j$.

Hence, the possibility for each player in S_j to improve its utility is as below:

$$\begin{aligned} u_1(c^1(\mu)) \geq u_1(c^1(\tau)) &\iff \gamma \geq \frac{1}{2}(\varepsilon_j - a_i) \\ u_{2j}(c^{2j}(\mu)) \geq u_{2j}(c^{2j}(\tau)) &\iff \varepsilon_j \geq \frac{1}{2}\delta_j \\ u_3(c^3(\mu)) \geq u_3(c^3(\tau)) &\iff \delta_j \geq b_i + \frac{1}{2}\gamma \end{aligned} \tag{27}$$

It thus follows that S_j improves upon τ if and only if all of the inequalities in (27) [7] hold with at least one inequality holding strictly, provided that μ indeed become a transfer among $\mathbb{N}$ with these parameter values of $\varepsilon_j, \delta_j, \gamma$.

We now look for parameter values which satisfy (27) and make μ a transfer among $\mathbb{N}$. Let $\gamma, \varepsilon_j, \delta_j$ be parameters satisfying (27) with at least one inequality holding strictly.

[7] From now on, we simply refer to (27) instead of referring to "the inequalities in (27)" to avoid combersome repetition.

Then, starting from γ satisfying a sequence of inequalities in (27), one obtains

$$\begin{aligned}\gamma &\geq \frac{1}{2}(\varepsilon_j - a_i) \\ &\geq \frac{1}{4}\delta_j - \frac{1}{2}a_i \\ &\geq \frac{1}{8}\gamma + \frac{1}{4}b_i - \frac{1}{2}a_i\end{aligned}$$

where at least one out of three inequalities holds strictly for S_j to improve upon τ. It follows that

$$\gamma > \frac{2}{7}(b_i - 2a_i) \tag{28}$$

Starting from each of the remaining ε_j or δ_j satsifying a sequence of inequalities in (27), one obtains exactly in the similar manner as above

$$\varepsilon_j > \frac{1}{7}(4b_i - a_i) \tag{29}$$

$$\delta_j > \frac{2}{7}(4b_i - a_i) \tag{30}$$

Therefore, S_j can improve upon τ if and only if there exist $\gamma \in [-c, 1-c]$, $\varepsilon_j \in [-a_j, 1-a_j]$, $\delta_j \in [-b_j, \frac{1}{2} - b_j]$ satisfying the inequalities (28), (29), (30).

It thus follows that S_j cannot improve upon τ if and only if for any $\gamma \in [-c, 1-c]$, $\varepsilon_j \in [-a_j, 1-a_j]$, $\delta_j \in [-b_j, \frac{1}{2} - b_j]$ at least one of the inequalities (28), (29), (30) is violated. Now, for all $\gamma \in [-c, 1-c]$ to violate (28) it is necessary and sufficient that the inequality

$$1 - c \leq \frac{2}{7}(b_i - 2a_i) \tag{31}$$

holds. Similarly, for all $\varepsilon_j \in [-a_j, 1-a_j]$ or for all $\delta_j \in [-b_j, \frac{1}{2} - b_j]$ to violate (29) or (30) respectively, it is necessary and sufficient that the following respective inequality

$$1 - a_j \leq \frac{1}{7}(4b_i - a_i) \tag{32}$$

$$\frac{1}{2} - b_j \leq \frac{2}{7}(4b_i - a_i) \tag{33}$$

holds for $i, j = 1, 2,\ i \neq j$. We rewrite the conditions (31), (32), (33), respectively, as below:

$$\frac{1}{7}(2a_i - b_i) \leq \frac{1}{2}(c - 1) \tag{34}$$

$$\frac{1}{7}(2a_i - b_i) \leq \frac{1}{4}(a - 1) \tag{35}$$

$$\frac{1}{7}(2a_i - b_i) \leq b - \frac{1}{2} \tag{36}$$

for $i = 1, 2$.

To sum up, S_j cannot improve upon τ if and only if at least one of the inequalities (34), (35), (36) is satisfied, or simply, if and only if the following condition holds for $i = 1, 2$:

$$\frac{1}{7}(2a_i - b_i) \leq \max\left\{\frac{1}{4}(a-1)\ ,\ b - \frac{1}{2}\ ,\ \frac{1}{2}(c-1)\right\}\ . \tag{37}$$

Therefore, one obtains the following proposition characterizing the core transfers in a Wicksellian transfer game for $\mathbb{N} = \{1, 2_1, 2_2, 3\}$.

Proposition 3.2 *If a transfer* $\tau = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \tau_3) \in [0,1]^5$ *among* $\mathbb{N} = \{1, 2_1, 2_2, 3\}$ *is a core transfer, then it satisfies the inequalities*

$$\frac{1}{7}(2\tau_{1i} - \tau_{2i}) \leq \max\left\{\frac{1}{4}(\tau_1 - 1)\ ,\ \tau_2 - \frac{1}{2}\ ,\ \frac{1}{2}(\tau_3 - 1)\right\} \tag{38}$$

for $i = 1, 2$.

Let $\tau = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \tau_3) = (a_1, a_2, b_1, b_2, c)$ be a core transfer among $\mathbb{N} = \{1, 2_1, 2_2, 3\}$. Then, it is individually rational and Pareto optimal. By the IR condition (23) the lefthand side of (34), (35), (36) is nonnegative, and at least one of the equalities $a = 1, b = 1, c = 1$ holds by the Pareto optimality. Let us consider the following two cases of values taken by components of the transfer.

Case I: $0 < b \leq \dfrac{1}{2}$

In this case we have $a = 1$ or $c = 1$ by the Pareto optimality and thus, the righthand side of (37) is 0 for each $i = 1, 2$. Hence, we have

$$a_i = \frac{1}{2}b_i \text{ for } i = 1, 2, \text{ and } a = \frac{1}{2}b\ . \tag{39}$$

It thus follows that we have $c = 1$. Therefore, by taking account of the IR condition (23), one has

$$b = \frac{1}{2}\ , \text{ and } a = \frac{1}{4}\ . \tag{40}$$

Case II: $\dfrac{1}{2} < b \leq 1$

In this case for any $i = 1, 2$ the right hand side of (37) is

$$\max\left\{\frac{1}{4}(a-1)\ ,\ b - \frac{1}{2}\ ,\ \frac{1}{2}(c-1)\right\} = b - \frac{1}{2} > 0\ . \tag{41}$$

Therefore, if $S_i \subset \mathbb{N}$ cannot improve upon τ for any $i = 1, 2$, then one has

$$(\forall i = 1, 2)\ 0 \leq \frac{1}{7}(2a_i - b_i) \leq b - \frac{1}{2}\ , \tag{42}$$

and by summing over i

$$0 \leq \frac{1}{7}(2a - b) \leq 2b - \frac{1}{2}\ . \tag{43}$$

Now, if $a < 1$ and $c < 1$, by the Pareto optimality we have $b = 1$ which means (42) is satisfied. It thus follows from the IR condition that we have

$$\begin{gathered}(\forall i = 1,2)\ \frac{1}{4} \le a_i < \frac{3}{4} \ \text{ and }\ b_i = \frac{1}{2}\ , \\ \frac{1}{2} \le a < 1 \ \text{ and }\ \frac{1}{2}a \le c < 1\ . \end{gathered} \tag{44}$$

When $a = 1$, it follows from (43) that $b \ge 3/5$ so that by taking account of the IR conditions one obtains:

$$a = 1\ ,\ \frac{3}{5} \le b \le 1\ ,\ \frac{1}{2} \le c \le 1\ . \tag{45}$$

On the other hand, when $c = 1$, by consideration of (37), one has

$$a_i = \frac{1}{2}b_i \text{ for } i = 1,2\ ,\ \frac{1}{4} \le a \le \frac{1}{2}\ ,\ \frac{1}{2} \le b \le 1\ ,\ c = 1\ . \tag{46}$$

Since the result of Case I can be combined to that of Case IIB, we can summarize the above results as follows:

$$\begin{aligned} &\text{(i) } (\forall i = 1,2)\ \frac{1}{4} \le a_i < \frac{3}{4} \ \text{ and }\ b_i = \frac{1}{2}\ , \\ &\qquad \frac{1}{2} \le a < 1 \ \text{ and }\ \frac{1}{2}a \le c < 1\ . \\ &\text{(ii) } \quad a_i = \frac{1}{2}b_i \text{ for } i = 1,2\ , \\ &\qquad \frac{1}{4} \le a \le \frac{1}{2}\ ,\ \frac{1}{2} \le b \le 1\ ,\ c = 1\ . \end{aligned}$$

We thus obtained the following:

Proposition 3.3 *Given any core transfer* $\tau = (\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \tau_3) \in [0,1]^5$ *among* $\mathbb{N} = \{1, 2_1, 2_2, 3\}$*, it satisfies one of the conditions below:*

$$\begin{aligned} &\text{(i) } (\forall i = 1,2)\ \frac{1}{4} \le \tau_{1i} < \frac{3}{4} \ \textit{ and }\ \tau_{2i} = \frac{1}{2}\ , \\ &\qquad \frac{1}{2} \le \tau_1 < 1 \ \textit{ and }\ \frac{1}{2}\tau_1 \le \tau_3 < 1\ . \\ &\text{(ii) } \quad \tau_{1i} = \frac{1}{2}\tau_{2i} \textit{ for } i = 1,2\ , \\ &\qquad \frac{1}{4} \le \tau_1 \le \frac{1}{2}\ ,\ \frac{1}{2} \le \tau_2 \le 1\ ,\ \tau_3 = 1\ . \end{aligned} \tag{47}$$

4 Wicksellian Transfer Games with Increasing Competitors

4.1 A Transfer Game with n Competing Type 2 Players

For the purpose of analizing how the monopolistic or monopsonistic power of type 1 or type 3 player might be affected, possibly for their favor, by an increasing number of type 2 players, each of whom faces, so to speak, progressively tough competiton among their own types, transfer games with n type 2 players along with single type 1 and type 3 players are considered in this section. Both of type 1 and type 3 players enjoy monopolistic and monopsonistic powers whereas n type 2 players face competition among players of the same type.

Thus, the set of all players is given by

$$\mathbb{N} = \{1, 2_1, \ldots, 2_n, 3\}.$$

The utility functions, u_1, u_3, and the initial endowments vectors, e_1, e_3, of players 1 and 3 are same as the ones in the basic model of the previous section. As to type 2 players $2_1, \ldots, 2_n$, their utility functions, $u_{21}, \ldots, u_{2n}$ are same as that of the player 2's utility function u_2, but their initial endowments vectors, $e_{21}, \ldots, e_{2n}$, are given so as to make the total endowments as a whole remain unchanged; thus,

$$u_{21} = \cdots = u_{2n} = u_2,\ e_{21} = \cdots = e_{2n} = \frac{1}{n} e_2$$

where the suffix $2j$ of utility functions or initial endowments vectors indicates an associated respective player 2_j. In a transfer game with the set of players $\mathbb{N}$, the system of transfers of commodities/assets is composed of n Wicksellian triangles given by $\Delta 12_j3,\ j = 1, \ldots, n$.

A transfer τ among $\mathbb{N} = \{1, 2_1, \ldots, 2_n, 3\}$ is now denoted by

$$\tau = (\tau_{11}, \ldots, \tau_{1n}, \tau_{21}, \ldots, \tau_{2n}, \tau_3)$$

where i in the suffix of τ_{ij} shows the *sender* of the transfer. For $i = 1$, j shows the *receiver* 2_j, and for $i = 2$, j shows the *sender* 2_j. As in the previous section, we use for convenience the following notation:

$$\tau_i = \sum_{j=1}^{n} \tau_{ij} \quad \text{for } i = 1, 2\ .$$

In this section $\tau = (\tau_{11}, \ldots, \tau_{1n}, \tau_{21}, \ldots, \tau_{2n}, \tau_3) \in [0, 1]^{2n+1}$ is called a *transfer* among $\mathbb{N}$ if it satisfies the constraint

$$0 \leq \tau_1 \leq 1, \quad 0 \leq \tau_{2j} \leq \frac{1}{n} \text{ for } j = 1, \ldots, n$$

that is, a transfer must be done within the obvious constraint given by each player's initial endowments. The Figure 5 shows a transfer composed of n Wicksellian triangles.

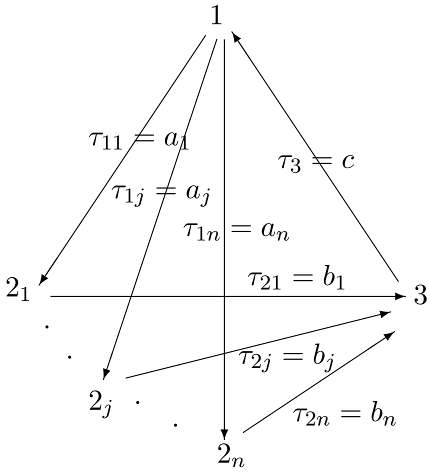

Figure 5: A Transfer $\tau = (\tau_{11}, \ldots, \tau_{1n}, \tau_{21}, \ldots, \tau_{2n}, \tau_3) = (a_1, \ldots, a_n, b_1, \ldots, b_n, c)$ using n Competing Wiksellian Triangles $\Delta 12_j 3$, $j = 1, \ldots, n$.

4.2 Pareto Optimal Transfers in a Wicksellian Transfer Game with n Competing Type 2 Players

For any subset $T \subset \{2_1, \ldots, 2_n\}$ of type 2 players, the subset S_T is defined by

$$S_T = \{1, 3\} \cup T \quad (= \cup_{j \in T} \Delta 12_j 3) \ .$$

It is clearly understood from the arguments in the previous section that a transfer τ among $\mathbb{N}$ is a core transfer if and only if it is individually rational and no subsets $S \subset \mathbb{N}$ containing at least one Wicksellian triangle, so that $\Delta 12_j 3 \subset S$ for some $j = 1, \ldots, n$, can improve upon it.

The IR condition for each player in $\mathbb{N}$ is given as follows:

$$\begin{aligned} u_1(c^1(\tau)) &= 1 - \tau_1 + 2\tau_3 \geq 1 \\ u_{2j}(c^{2j}(\tau)) &= \frac{1}{n} - \tau_{2j} + 2\tau_{1j} \geq \frac{1}{n}, \ j = 1, \ldots, n \\ u_3(c^3(\tau)) &= 1 - \tau_3 + 2\tau_2 \geq 1 \end{aligned}$$

Thus, the IR condition for each player in $\mathbb{N}$ becomes

$$\tau_3 \geq \frac{1}{2}\tau_1 \ , \quad \tau_{1j} \geq \frac{1}{2}\tau_{2j}, \ j = 1, \ldots, n \ , \quad \tau_2 \geq \frac{1}{2}\tau_3 \ . \tag{48}$$

Given a transfer τ among $\mathbb{N} = \{1, 2_1, \ldots, 2_n, 3\}$ and a set S_T for a nonempty subset $T \subset \{2_1, \ldots, 2_n\}$, we now derive conditions under which τ cannot be improved upon by S_T. In this section T' is the complement of T in $\{2_1, \ldots, 2_n\}$ so that $T' = \{2_1, \ldots, 2_n\} \setminus T$. Let us proceed our arguments in this section by slightly simplifying the notation for the transfer τ as in the previous section by writing

$$\begin{aligned} \tau &= (\tau_{11}, \ldots, \tau_{1n}, \tau_{21}, \ldots, \tau_{2n}, \tau_3) \\ &= (a_1, \ldots, a_n, b_1, \ldots, b_n, c) \ . \end{aligned} \tag{49}$$

Then define $\mu = (\mu_{11}, \ldots, \mu_{1n}, \mu_{21}, \ldots, \mu_{2n}, \mu_3) \in [0,1]^{2n+1}$ by

$$\begin{aligned} (\forall j \in T)\ \mu_{1j} &= a_j + \varepsilon_j, \quad \mu_{2j} = b_j + \delta_j\ , \\ (\forall k \in T')\ \mu_{1k} &= \mu_{2k} = 0\ , \\ \mu_3 &= c + \gamma \end{aligned} \tag{50}$$

where $\gamma, \varepsilon_j, \delta_j$ for $j \in T$ are real-valued parameters to be determined so as to make μ a transfer among $\mathbb{N}$. If μ is indeed a transfer among $\mathbb{N}$, it is feasible for S_T. Note that by a slight abuse of notation we may write $a_j, b_j, \varepsilon_j, \delta_j, \mu_{1j}$ or μ_{2j} for $j \in \{2_1, \ldots, 2_n\}$.

In order to examine whether a subset $S_T \subset \mathbb{N}$ can improve upon the transfer τ, let us compute the utility difference for each player in S_T between the consumption allocations induced by μ and τ. For some of their components the following notation is used:

$$a \equiv \sum_{j=1}^{n} a_j\ , a_T \equiv \sum_{j \in T} a_j\ , a_{T'} \equiv \sum_{j \in T'} a_j\ .$$

The notation $b, b_T, b_{T'}, \varepsilon_T, \delta_T$ is defined similarly with the convention that $a_{T'} = b_{T'} = 0$ whenever $T' = \emptyset$.

The utility difference for each player in S_T between the consumption allocations induced by μ and τ is given as follows:

$$\begin{aligned} u_1(c^1(\mu)) - u_1(c^1(\tau)) &= (1 - a_T - \varepsilon_T) + 2(c+\gamma) - (1 - a + 2c) \\ &= 2\gamma + (a_{T'} - \varepsilon_T) \\ u_{2j}(c^{2j}(\mu)) - u_{2j}(c^{2j}(\tau)) &= \left(\frac{1}{n} - b_j - \delta_j\right) + 2(a_j + \varepsilon_j) - \left(\frac{1}{n} - b_j + 2a_j\right) \\ &= 2\varepsilon_j - \delta_j \qquad \text{for each } j \in T \\ u_3(c^3(\mu)) - u_3(c^3(\tau)) &= (1 - c - \gamma) + 2(b_T + \delta_T) - (1 - c + 2b) \\ &= 2(\delta_T - b_{T'}) - \gamma \end{aligned}$$

Hence, the possibility for each player in S_T to improve its utility is as below:

$$\begin{aligned} u_1(c^1(\mu)) \geq u_1(c^1(\tau)) &\Longleftrightarrow \gamma \geq \frac{1}{2}(\varepsilon_T - a_{T'}) \\ u_{2j}(c^{2j}(\mu)) \geq u_{2j}(c^{2j}(\tau)) &\Longleftrightarrow \varepsilon_j \geq \frac{1}{2}\delta_j \qquad \text{for each } j \in T \\ u_3(c^3(\mu)) \geq u_3(c^3(\tau)) &\Longleftrightarrow \delta_T \geq b_{T'} + \frac{1}{2}\gamma \end{aligned} \tag{51}$$

It thus follows that S_T improves upon τ if and only if all of the inequalities in (51) [8] hold with at least one inequality holding strictly, provided that μ indeed become a transfer among $\mathbb{N}$ with these parameter values of $\varepsilon_j, \delta_j, \gamma$.

[8] From now on, we simply refer to (51) instead of referring to "the inequalities in (51)" to avoid combersome repetition.

We now shift our angle in viewing a possibility of S_T to improve upon τ, and look for parameter values which satisfy (51) and make μ a transfer among $\mathbb{N}$. Let parameters $\gamma, \varepsilon_T, \delta_T$ represent (or, rather summarize) parameters satisfying (51) with at least one inequality holding strictly. Then, starting from γ satisfying a sequence of inequalities in (51), one obtains

$$\begin{aligned}\gamma &\geq \frac{1}{2}\left(\varepsilon_T - a_{T'}\right)\\ &\geq \frac{1}{4}\delta_T - \frac{1}{2}a_{T'}\\ &\geq \frac{1}{8}\gamma + \frac{1}{4}b_{T'} - \frac{1}{2}a_{T'}\end{aligned}$$

where at least one out of three inequalities holds strictly for S_T to improve upon τ. It follows that

$$\gamma > \frac{2}{7}\left(b_{T'} - 2a_{T'}\right) \tag{52}$$

Starting from each of the remaining ε_T or δ_T satsifying a sequence of inequalities in (51), one obtains exactly in the similar manner as above

$$\varepsilon_T > \frac{1}{7}\left(4b_{T'} - a_{T'}\right) \tag{53}$$

$$\delta_T > \frac{2}{7}\left(4b_{T'} - a_{T'}\right) \tag{54}$$

Therefore, if S_T cannot improve upon τ, then for any $\gamma \in [-c, 1-c]$, $\varepsilon_T \in [-a_T, 1-a_T]$, $\delta_T \in [-b_T, \frac{\sharp T}{n} - b_T]$ at least one of the inequalities (52), (53), (54) is violated.

Now, for all $\gamma \in [-c, 1-c]$ to violate (52) it is necessary and sufficient that (see Figure 6) the inequality

$$1 - c \leq \frac{2}{7}(b_{T'} - 2a_{T'}) \tag{55}$$

holds. Similarly, for all $\varepsilon_T \in [-a_T, 1-a_T]$ or for all $\delta_T \in [-b_T, \frac{\sharp T}{n} - b_T]$ to violate (53) or (54) respectively, it is necessary and sufficient that the following respective inequality

$$1 - a_T \leq \frac{1}{7}\left(4b_{T'} - a_{T'}\right) \tag{56}$$

$$\frac{\sharp T}{n} - b_T \leq \frac{2}{7}\left(4b_{T'} - a_{T'}\right) \tag{57}$$

holds.

It will be convenient to rewrite the conditions (55), (56), (57) respectively as below:

$$\frac{1}{7}\left(2a_{T'} - b_{T'}\right) \leq \frac{1}{2}(c-1) \tag{58}$$

$$\frac{1}{7}\left(2a_{T'} - b_{T'}\right) \leq \frac{1}{4}\left(a-1\right) \tag{59}$$

$$\frac{1}{7}\left(2a_{T'} - b_{T'}\right) \leq b - \frac{\sharp T}{n} \tag{60}$$

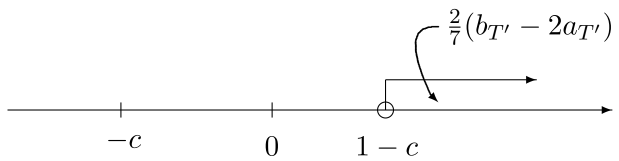

Figure 6: The Region of Parameter Values

To sum up, if S_T cannot improve upon τ, then at least one of the inequalities (58), (59), (60) is satisfied, or simply, the following condition holds:

$$\frac{1}{7}\left(2a_{T'} - b_{T'}\right) \leq \max\left\{\frac{1}{2}(c-1), \frac{1}{4}\left(a-1\right), b - \frac{\sharp T}{n}\right\} \tag{61}$$

In case of $S_T = \mathbb{N}$, $T = \{2_1, \ldots, 2_n\}$. Hence, $T' = \emptyset$, which means both the left and the right hand sides of the inequality in each of (58), (59), (60) are 0, and the right hand side of (60) is $b-1$. We thus obtain a necessary and sufficient condition for a transfer τ among $\mathbb{N}$ to be Pareto optimal.

Theorem 4.1 *A transfer $\tau = (\tau_{11}, \ldots, \tau_{1n}, \tau_{21}, \ldots, \tau_{2n}, \tau_3)$ among $\mathbb{N} = \{1, 2_1, \ldots, 2_n, 3\}$ is Pareto optimal if and only if $\tau_i = 1$ for some $i = 1, 2, 3$.*

In the next subsection we turn to a necessary and sufficient condition for a transfer τ among $\mathbb{N}$ to be a core transfer.

4.3 The Core of Wicksellian Transfer Games with Increasing Type 2 Players

As is pointed out earlier in Section 3 a transfer $\tau = (\tau_{11}, \ldots, \tau_{1n}, \tau_{21}, \ldots, \tau_{2n}, \tau_3)$ among $\mathbb{N}$ cannot be improved upon by any subset of $\mathbb{N}$ unless it contains a Wicksellian triangle, or at least one of each three types of players. Thus it follows from the arguments in the previous subsection that a transfer τ among $\mathbb{N}$ is a core transfer if and only if it is individually rational, Pareto optimal, and satisfies the inequality (61) for all $T \subset \{2_1, \ldots, 2_n\}$ for which $1 \leq \sharp T \leq n-1$. Therefore, one obtains the following theorem characterizing the core transfers in a Wicksellian transfer game.

Theorem 4.2 *If a transfer $\tau = (\tau_{11}, \ldots, \tau_{1n}, \tau_{21}, \ldots, \tau_{2n}, \tau_3) = (a_1, \ldots, a_n, b_1, \ldots, b_n, c)$ among $\mathbb{N}$ is a core transfer, then it is individually rational, Pareto optimal, and satisfies the inequality*

$$\frac{1}{7}\left(2a_{T'} - b_{T'}\right) \leq \max\left\{\frac{1}{2}(c-1), \frac{1}{4}\left(a-1\right), b - \frac{\sharp T}{n}\right\} \tag{62}$$

for any subset $S_T \subset \mathbb{N}$ with $1 \leq \sharp T \leq n-1$.

Henceforth in this section, by examining transfers among $\mathbb{N}$ that are both individually rational and Pareto optimal, we substantiate the condition given by (62).

Let τ, whose components are written as in (49), be a transfer among $\mathbb{N}$, which is individually rational and Pareto optimal. Note first that by the IR condition (48) the lefthand side of (62) is nonnegative, and at least one of the equalities $a = 1, b = 1, c = 1$ holds by the Pareto optimality condition. We consider the following two cases of values taken by components of the transfer.

Case I: $0 < b \leq 1 - \dfrac{1}{n}$

In this case we have $a = 1$ or $c = 1$ by the Pareto optimality and thus, the righthand side of (62) is 0 for T with $\sharp T = n - 1$. Hence, by taking the corresponding subset $T = \{2_1, \ldots, 2_n\} \setminus \{i\}$ for any $i = 1, \ldots, n$, one obtains

$$(\forall i = 1, \ldots, n)\ a_i = \frac{1}{2} b_i\ . \tag{63}$$

It follows that one has

$$a = \sum_i a_i = \frac{1}{2} \sum_i b_i = \frac{1}{2} b \leq \frac{1}{2} \left(1 - \frac{1}{n} \right)\ . \tag{64}$$

Hence, in this case $c = 1$. Therefore, by taking account of the IR conditions,

$$\frac{1}{2} \leq b \leq 1 - \frac{1}{n} \tag{65}$$

$$\frac{1}{4} \leq a = \frac{1}{2} b \leq \frac{1}{2} \left(1 - \frac{1}{n} \right)\ . \tag{66}$$

The consideration of other sets T with $1 \leq \sharp T \leq n - 1$ does not tighten the right hand side of (65) as (63) implies the left hand side of (62) must be 0 for any T with $1 \leq \sharp T \leq n - 1$.

Case II: $1 \geq b > 1 - \dfrac{1}{n}$

In this case for any T with $1 \leq \sharp T \leq n - 1$, the right hand side of (62) is

$$\max \left\{ \frac{1}{2}(c-1), \frac{1}{4}(a-1), b - \frac{\sharp T}{n} \right\} = b - \frac{\sharp T}{n} \geq b - \left(1 - \frac{1}{n} \right) > 0\ . \tag{67}$$

Therefore, if $S_T \subset \mathbb{N}$ cannot improve upon τ for any T with $\sharp T = n - 1$, then it cannot improve upon τ for any T with $1 \leq \sharp T \leq n - 1$.Applying (62) to $\sharp T = n - 1$ with $T = \{1, \ldots, n\} \setminus \{i\}$, one obtains

$$(\forall i = 1, \ldots, n)\ 0 \leq \frac{1}{7}(2a_i - b_i) \leq b - \left(1 - \frac{1}{n} \right)\ . \tag{68}$$

Let $m \in \{1, \ldots, n\}$ be such that $2a_m - b_m = \max \{2a_i - b_i \mid\ i = 1, \ldots, n\}$. It follows from (68) that

$$0 \leq \frac{1}{7} \left[\frac{1}{n}(2a - b) \right] \leq \frac{1}{7}(2a_m - b_m) \leq b - \left(1 - \frac{1}{n} \right)\ . \tag{69}$$

Case IIA: $a < 1$ and $c < 1$

By the Pareto optimality we have $b = 1$, and thus, $(\forall i = 1, \dots, n)\ b_i = 1/n$. It then follows from (68) that

$$(\forall i = 1, \dots, n)\ \frac{1}{2n} \leq a_i \leq \frac{4}{n}\ . \tag{70}$$

Considering the lower bound for a_i in (70), its upper bound is also given by

$$(\forall i = 1, \dots, n)\ a_i \leq 1 - \frac{n-1}{2n}\ .$$

Therfore, by taking account of the IR condition of type 1 player, we obtain:

$$\begin{aligned} (\forall i = 1, \dots, n)\ \frac{1}{2n} \leq a_i < \min\left\{\frac{4}{n}, 1 - \frac{n-1}{2n}\right\}, \\ b_i = \frac{1}{n}, \quad \text{and} \quad \frac{1}{4} \leq \frac{1}{2}a \leq c\ . \end{aligned} \tag{71}$$

Case IIB: $a = 1$ or $c = 1$

In case $a = 1$, it follows from (69) and the IR conditions of player 1 and player 3 that

$$\frac{1}{2}c \leq b \leq 1 - \frac{6}{7n+1}\ , \quad \frac{1}{2} \leq c\ . \tag{72}$$

In case $c = 1$, (68) does not give any restrictions on the value of a and b.

Let us summarize the above results of Case I and Case II using the range of values taken by b:

(i) $b = 1$

$$(\forall i = 1, \dots, n)\ \frac{1}{2n} \leq a_i < \min\left\{\frac{4}{n}, 1 - \frac{n-1}{2n}\right\},$$

$$b_i = \frac{1}{n}, \quad \text{and} \quad \frac{1}{4} \leq \frac{1}{2}a \leq c\ .$$

(ii) $1 - \frac{1}{n} < b < 1$

$$\cdot\ a = 1\ , \quad \frac{1}{2}c \leq b \leq 1 - \frac{6}{7n+1}\ , \quad \frac{1}{2} \leq c\ ,$$

$$\cdot\ c = 1, \quad \text{and} \quad \frac{1}{2}b \leq a\ .$$

(iii) $0 < b \leq 1 - \frac{1}{n}$

$$(\forall i = 1, \dots, n)\ a_i = \frac{1}{2}b_i\ , \quad \frac{1}{4} \leq a\ , \quad \text{and} \quad c = 1\ .$$

It follows from the above arguments that we have established the following (see Figure 7):

Theorem 4.3 *Given any core transfer* $\tau = (\tau_{11}, \ldots, \tau_{1n}, \tau_{21}, \ldots, \tau_{2n}, \tau_3)$ *among* $\mathbb{N}$, *it satisfies one of the conditions below:*

$$
\begin{aligned}
&\text{(i) } \textit{For } \ \frac{1}{2} \le \tau_2 \le 1 - \frac{1}{n} \\
&\quad (\forall i = 1, \ldots, n)\ \tau_{1i} = \frac{1}{2}\tau_{2i}\ , \quad \textit{and} \ \ \tau_3 = 1\ . \\
&\text{(ii) } \textit{For } \ 1 - \frac{1}{n} < \tau_2 < 1 \\
&\quad \cdot\, \tau_1 = 1\ , \quad \frac{1}{2}\tau_3 \le \tau_2 \le 1 - \frac{6}{7n+1}\ , \quad \frac{1}{2} \le \tau_3\ , \\
&\quad \cdot\, \tau_3 = 1, \quad \textit{and} \ \ \frac{1}{2}\tau_2 \le \tau_1\ . \\
&\text{(iii) } \textit{For } \ \tau_2 = 1 \\
&\quad (\forall i = 1, \ldots, n)\ \frac{1}{2n} \le \tau_{1i} < \min\left\{\frac{4}{n}, 1 - \frac{n-1}{2n}\right\}, \\
&\quad \tau_{2i} = \frac{1}{n}, \quad \textit{and} \ \ \frac{1}{4} \le \frac{1}{2}\tau_1 \le \tau_3\ .
\end{aligned}
\tag{73}
$$

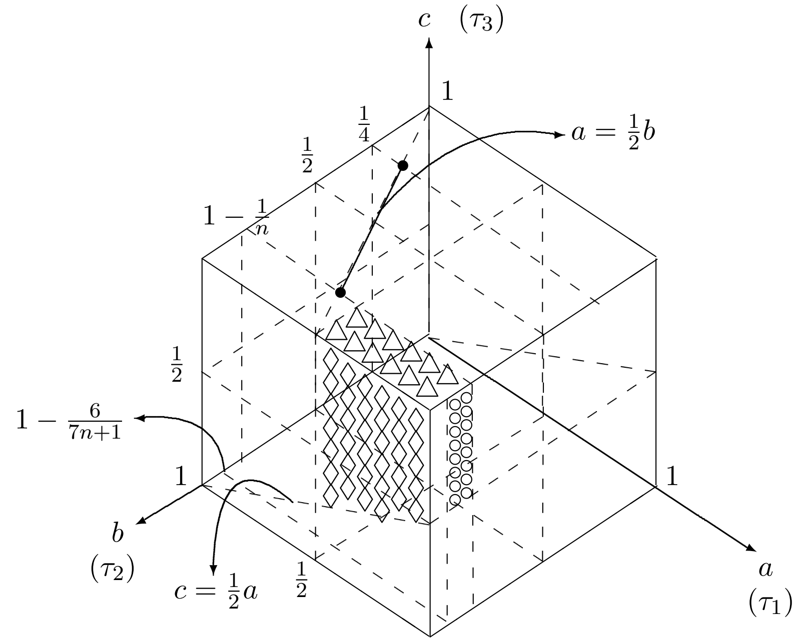

Figure 7: The Region of Parameter Values for Core Transfers in $\mathbb{N} = \{1, 2_1, \ldots, 2_n, 3\}$

5 Discussions of the Main Result

It would be instructive to give some interpretation and discussion of the result of the main theorem. The main result clarifies the essential nature of core transfers in the Wicksellian trasfer game. Let $\tau = (\tau_{11}, \ldots, \tau_{1n}, \tau_{21}, \ldots, \tau_{2n}, \tau_3) = (a_1, \ldots, a_n, b_1, \ldots, b_n, c)$ be a core transfer. Following our proof in Section 4.2, we classify regions of parameter values using that of type 2 players' transfer values.

Case I ($0 < b \leq 1 - \frac{1}{n}$) represents the case in which the total amount of transfer by type 2 players does not exceed the maximal possible amount that could be sent by all type 2 players but one. In such a case Player 3 does not exercise any monopsonistic power as the buyer of commodity 2 held only by type 2 players. The total amount a of transfers from Player 1 to type 2 players and the total amount b of transfers from type 2 players to Player 3 must satisfy (65) and (66). Let us closely look at these conditions in comparison to those in Proposition 2.2 and Proposition 3.3.

An addition of one type 2 player to the basic Wicksellian transfer game produces a significant effect on core transfers. In a case in which the total amount of transfer by type 2 players does not exceed the maximal possible amount that could be sent by all type 2 players but one, with two type 2 competitors, a core transfer must be such that $a = 1/4$, $b = 1/2$ and $c = 1$, that is the best case for Player 1 but the worst one for Player 3. Hence, a natural question might arise: An increase in the number of type 2 players, who are 'buyers' to Player 1 and 'sellers' to Player 3, does only benefit Player 1 as a 'monopolistic seller'? (65) and (66) answer negatively to this question: In fact as the number n of type 2 players increases, the maximal amount of transfer that Player 3 would receive converges to 1. In a sense Player 3 can regain the power of a monopsonist as in the basic Wicksellian game when the number n of player 2 increases, and Player 1's monopolistic power is affected adversely as the total amount b that is received by Player 3 increases.

Note that as long as the total amount of transfer by type 2 players does not exceed the maximal possible amount that could be sent by all type 2 players but one, the utility level of type 2 players always remain at the same level as the one without any transfers among players whenever the number of type 2 players increases to more than two. This might be regarded as *de facto* manifestation of *perfect competition* among type 2 players as in the case of price-taking markets.

One may also note that this is the case in which Player 1 exercises the most power as a monopsonist and faces a subtle situation as a monopolist when the number of type 2 players increases. Player 1 benefits most when the number of type 2 players is two since it receives a transfer of all endowments of Player 3 whereas Player 1 transfers only the minimum possible amount of its endowments to type 2 players. But as the number n of type 2 players increases, there is a possibility of transferring its endowments by the amount exceeding the minimum amount of 1/4 depending upon the total amount b of transfer by type 2 players but it transfers at most one-half of its endowments.

Case II ($1 \geq b > 1-\frac{1}{n}$) represents the case in which the total amount of transfer by type 2 players exceeds the maximal possible amount that could be sent by all type 2 players but one. A core transfer must satisfy (69). Let us examine three terms in (69).

Observe first a sort of 'magic number' $\frac{1}{7}$. A unit of type 2 commodity sent to Player 3 is worth MRS^3_{23} units of type 3 commodity which, when sent to Player 1 is worth $\mathrm{MRS}^3_{23}\mathrm{MRS}^1_{31}$ units of type 1 commodity which, when sent to type 2 players is worth $\mathrm{MRS}^3_{23}\mathrm{MRS}^1_{31}\mathrm{MRS}^2_{12}(= 8)$ units of type 2 commodity in utility terms so that when type 2 players transfer 1 unit of their endowments their net maximal gain in utility terms could be worth $7 = \mathrm{MRS}^3_{23}\mathrm{MRS}^1_{31}\mathrm{MRS}^2_{12} - 1$ units of their own endowments. This means that in order for type 2 players to gain a unit of their utility, a minimal amount of their endowments needed to be transferred to Player 3 is given by this magic number $\frac{1}{7}$ which represents the amount $\frac{1}{\mathrm{MRS}^3_{23}\mathrm{MRS}^1_{31}\mathrm{MRS}^2_{12} - 1}$.

The first term in (69) represents the minimal amount of type 2 endowments needed to be transferred to Player 3 for the average net utility to be gained by type 2 players by the transfer. Its second term represents the minimal amount of type 2 endowments needed to be transferred to Player 3 for the maximal net utility gained by a type 2 individual player by the transfer. Its third term represents the excess of the amount transferred by all type 2 players to Player 3 over the maximal possible amount of endowments of all type 2 players but one.

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