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Akira Yamazaki
(Meisei University)

Hodokubo 2-1-1, Hino, Tokyo 191-8506
School of Economics, Meisei University

Phone: 042-591-9479 Fax: 042-599-3024
URL: http://keizai.meisei-u.ac.jp/econ/
E-mail: keizai@econ.meisei-u.ac.jp

On the Perception and Representation of Economic Quantity in the History of Economic Analysis in view of the Debreu Conjecture*

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1 Introduction

General equilibrium theory initiated by L.Walras [18] provides a solid theoretical framework of modern economic analysis. From purely theoretical point of view its fundamental mathematical structure is established throughout a series of papers in 1950-60's in which the existence of an equilibrium is established.

The purpose of this paper is to provide an interpretation of perception and representation of economic quantities and economic variables, specifically relating to the concept of demand as their representative, in the fundamental theoretical framework from the point of view of the Debreu [7] conjecture.

1.1 Debreu Conjecture

Gerard Debreu in his presidential address to the Econometric Society proffered the following conjecture to the members of the society:

> One expects that if the measure ν is suitably diffused over the space A (of economic agents' characteristics), integration over A

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of the demand *correspondences* of the agents will yield a total *demand function*, possibly even a total demand function of class C^1.

His talk in this address was later published as a paper titled "Smooth Preferences"(Debreu [8, p.614]). In his paper Debreu introduced a differentiable structure in a space of preference relations, and clarified conditions under which demand functions are differentiable . His conjecture above posed economic theorists a question that if, by considering a distribution over preference relations and initial endowments of economic agents who constitute an economy, it is "diffused" in some appropriate sense so that there is a sufficient variety of characteristics of economic agents or consumers, then it might possibly be the case that total demand becomes a function or even a continuously differentiable function notwithstanding individual demands being correspondences.

The Debreu conjecture gave a substantial impact on researches centered around the University of California, Berkley in 1970-80's. Since the significance of this conjecture is essentially related to the perception and representation of economic quantity, the purpose of this paper is to review the history of economic analysis from the Debreu conjecture's view point.

But, first, we would like to confirm theoretical framework in which the conjecture is posed.

1.2 Related Theoretical Framework

The discussion of this paper presupposes the theoretical framework of general equilibrium theory that has been established through a series of work in 1950-60's. Basic components of the fundamental model representing this framework are as follows:

1. a population of economic agents;
2. a set of commodities;
3. prices of commodities;
4. a market equilibrium.

Each of these factors assumes the following mathematical representation: A population of economic agents is given by a finite set or an atomless measure space, i.e. a continuum of economic agents. A set of commodities or the commodity space is given by an ℓ-dimensional Euclidean space $\mathbb{R}^\ell$ if the number of commodities is given by a positive natural number ℓ, and by a linear topological space L if the number of commodities is not finite. Prices of commodities are represented by the dual space of the basic commodity space. In addition, economic variables that form a market equilibrium are total demand correspondence (or function) and total supply correspondence (or function).

The basis of the model is commodities that form the commodity space and their quantitative representation. One must be clear about how different kinds of commodities are distinguished and how their quantities are expressed. Debreu [8] discusses most explicitly the way in which commodities must be distinguished. He distinguishes commodities according to (1) their physical characteristics, (2) dates at which they are available, (3) locations where they are available, and (4) events at which they are available ([7, Chapter 2 and Chapter 7]). It has become the standard way of distinguishing commodities in the literature of economic theory since then.

Quantities of particular commodities thus distinguished are represented by "any real numbers" without farther mathematical restrictions. Hence, as explained earlier, the commodity space is given by an ℓ-dimensional Euclidean space $\mathbb{R}^\ell$ when the number of commodities is limited to a finite natural number ℓ.

1.3 Perception of Economic Quantity

Notwithstanding the fact that economic quantity has taken bo be represented by any real number in the standard theoretical literature since 1950's, there has been a different percepting view on economic quantity itself. For instance even in the book of Debreu [7, p.30], one can find a statement expressing such a view.

In his case he does not have any objections to using any real number in representing quantity of commodities such as wheat or those that are liquid. But he admits that there are commodities such as trucks, a quantity of which must be an integer. Here is a quotation from Debreu [7, p.30].

> The quantity of certain kind of wheat is expressed by a number

> of bushels which can satisfactorily be assumed to be any (non-negative) real number.
>
> A quantity of certain kind of liquid such as gasoline is expressed by a number of litters or gallons which can be assumed to be any (non-negative) real number.
>
> A quantity of well-defined trucks is an integer; but it will be assumed instead that this quantity can be any real number.

Despite his perception of such an economic quantity, in his theoretical analysis he goes ahead to take the commodity space as $\mathbb{R}^\ell$, and admits economic transactions of real number units of various commodities.

As one sees in his "pretext" quoted below, it is merely for the purpose of seeking analytical simplification.[1)]

> This assumption of *perfect divisibility is imposed by the present stage of development of economics*; it is quite acceptable for an economic agent producing or consuming a large number of trucks. Similar goods are machine tools, linotypes, cranes, Bessemer converters, houses,...[2)]

The "perfect divisibility" in the quotation is in the sense of economics that is different from "divisibility" in the sense of mathematics.[3)]

In this paper we do not discuss the representation of prices that reflect the valuation of commodities. In the next section we would like to follow a path in the history of economic analysis concerning the perception and the representation of economic quantity.

[1)] In the latter part of the quotation, he does not give an explicit explanation of why the assumption of perfect divisibility of commodities is acceptable for an economic agent transacting a large number of commodities. I would presume that Debreu's perception in this regard is similar to that of Pareto or Walras, which we take up in the later part of this paper

[2)] My italics.

[3)] In general, "divisibility" in arithmetic means that a rational number can be expressed by a finite digits. Thus, as an analytical concept, divisibility in the sense of economics stands in polar to that in the sense of mathematics

2 Perception and Representation of Economic Quantity in the Representative Literature of Economic Analysis

In this section we shall take up the perception and representation of economic quantity as we find in the works of Augustin Cournot, Leon Walras, Vilfreto Pareto, and Alfred Marshall each of whose contributions to the economic analysis Joseph A. Schumpeter admits in his book [17].

2.1 Perception and Representation of Economic Quantity in Cournot

In establishing a theoretical foundation for explaining how exchange values of commodities are determined in markets, Cournot [4] proposed to clarify the concept of demand in markets by introducing an abstract mathematical concept of a function to represent market demand.

> Cournot [4, p. 37] "21. Admettons donc que le débit ou la demande annuelle D est, pour chaque denreé, une fonction particulière $F(p)$ du prix p de cette denreé. ......" [4)]

It appears to be there were no clear-cut distinctions, at the time when his book was written, among concepts such as demand, quantity demanded, supply, quantity supplied, equilibrium quantity, etc., which contributed to some confusion in the literature. [5)]

Demand Function and Continuity

By use of the mathematical concept of a function, Cournot perceived market demand as a demand function, probably for the first time in the literature of economic theory, and postulated the "continuity" property of the function at such an abstract level:

[4)] Its English translation is as follows. Cournot [5, p. 47] "Let us admit therefore that the annual sales or demand D is, for each article, a particular function $F(p)$ of the price p of such article. ......"

[5)] See Cournot [4, p.36, 20]. "En outre, qu'entend-on par la quantité demandé? Ce n'est sans doute pas celle qui se débite effectivement sur la demande des acheteurs; ......"

> Cournot[4, pp.38-39] "22. Nous admettrons que la fonction $F(p)$ qui exprime la loi de la demande ou du débit est une *fonction continue*, c'est-à-dire une fonction qui ne passe pas soudainement d'une valeur à une aurte, mais qui prend dans l'intervalle toutes les valeurs intermédiaires. *Il en pourrait être autrement si le nombre des consommateurs était très-limité:* ainsi, dans tel ménage, on pourra consommer précisément la même quantité de bois de chauffage, que le bois soit à 10 francs ou à 15 francs le stère; et l'on pourra réduire brusquement la consommation d'une quantité notable, si le prix du stère vient à dépasser cette dernière somme. " [6)]

The way Cournot explains the behavior of market demand in the above quotation is that changes in quantity demanded of a market demand function which represents "the law of demand or sale" [7)] occur in such a way that one value does not pass to another suddenly but by taking all intermediate values. In other words, he postulated the continuity property of a market demand function by requiring, so to speak, " the theorem of intermediate values" to hold.

This does not mean, however, that Cournot as a mathematician understood continuous functions in this way. My presumption is that he probably thought it would be easier to be understood by his contemporary economists if he rather explained continuity property by using one of the properties of

[6)] My italics. Its English tranlation is as follows: Cournot [4, pp.49-50] "22. We will assume that the function $F(p)$, which expresses the law of demand or of the market, is a *continuous function*, *i.e.* a function which does not pass suddenly from one value to another, but which takes in passing all intermediate values. *It might be otherwise if the number of consumers were very limited*: thus in a certain household the same quantity of firewood will possibly be used whether wood costs 10 francs or 15 francs the stere, and the consumption may suddenly be diminished if the price of the stere rises above the latter figure."

[7)] Cournot clearly states that he uses the word "the demand" (la demande) and "the sale"(le débit) synonymously. Cournot [4, pp.38-39] "...Le débit ou la demande (car pour nous ces deux mots sont synonymes, et nous ne voyons pas sous quel rappport la théorie aurait à tenir compte d'une demande qui n'est pas suivie de débit), le débit ou la demande, disons-nous, croît en général quand le prix décroît." Its English translation is as follows: Cournot [5, p.46] " ...The sales or the demand (for to us these two words are synonymous, and we do not see for what reason theory need take account of any demand which does not result in a sale)—the sales or the demand generally, we say, increases when the price decreases."

a continuous function in modern mathematics.

In fact, we could quote from one of his mathematics books the following:

> Cournot [6, p. 3] "Le caractère propre d'une fonction continue consiste en ce que l'on peu toujours assigner à l'une des variables des valeurs assez voisines pour que la différence entre les valeurs correspondantes de la fonction qui en dépend, tombe au-dessous de toute grandeurs donnée." [8)]

In this quotation the way he characterizes a continuous function can be understood to represent exactly the one using ε-δ in modern analysis or the one using neighborhoods in modern topological analysis.

Although Cournot pursues his analysis on the basis of what he called "the continuity" of a market demand function, he clearly admits, by way of an example, that the demand function might not well be continuous if the number of consumers is limited.

The ground for Cournot to reason individual demand functions to be discontinuous is not the indivisibility of commodities such that their quantitative representation should be limited to integers such as 1,2,.... The above example of Cournot does not exhibit a behavior of quantity demanded that declines slightly as its price moves up slightly, but that quantity demanded declines abruptly as its price increases to a certain level. If we are to express the Cournot's perception in terms of present day mathematical terms he would say that it is at most semi-continuous and might not deny its semi-continuity.[9)]

I would like to point out that it is possible to interpret the meaning of the "continuity" in the Cournot's book in a different sense. In fact, he went on to give further discussions on a continuous function in the same page that we quoted above:

> Cournot [4, p.39] "Si la fonction $F(p)$ est continue, elle jouira de la propriété commune à toutes les fonctions de cette nature, et

[8)] My English translation is as follows: "The proper characteristic of a continuous function consists in that one can always assign to each of the variables values sufficiently close so that the difference of the values corresponding to the function on which they depend, falls within for any given magnitudes."

[9)] In the quotation from Cournot he may be interpreted to deny the lower-semicontinuity of individual demand functions. We will further discuss on this aspect in Section 4 in addition to explaining the concept of semi-continuity.

sur laquelle reposent tant d'applications importantes de l'analyse mathématique: *les variations de la demande seront sensiblement proportionnelles aux variations du prix, tant que celles-ci seront de petites fractions du prix originaire.* D'ailleurs, ces variations seront de signes contraires, c'est-à-dire qu'à une augmentation de prix correspondra une diminution de la demande." [10)]

Here he explains the meaning of the continuity of the demand function $F_(p)$. In particular, the quotation in italics shows explicitly that the function $F_(p)$ allows "locally linear approximation". In other words, one can interpret that he in fact postulates the *differentiability* of the function in the name of the continuity of the function. We would like to take up this question again later with regard to the Debreu conjecture.

One of the reason why he had interests in the continuity of the demand function is that if the function as the object of main analysis is " continuous", it is very convenient for analytical purposes as analytical methods in mathematics can be applied.

Continuity by Aggregation Effects

Cournot thus admitting in general the discontinuity of individual demand functions on one hand, he claims that one should regard the market demand function to be continuous on the other. The most interesting part of his views with respect to his perception of economic variables is the following quotation that could be interpreted to be his recognition of the continuity of the market demand function which is induced by aggregation effect, the quotation of which follows the one earlier (Cournot [4, pp.38-39]).

> Cournot [4, p.39] "Mais plus le marché s'étendra, plus les combinaisons des besoins, des fortunes ou même des caprices, seront variées parmi les consommateurs, plus la fonction $F(p)$ approchera de varier avec p d'une manière continue. Si petite que soit la variation de p, il se trouvera des consommateurs placés dans une

[10)] Its English tranlation is as follows: Cournot [4, p.50] " If the function $F(p)$ is continuous, it will have the property common to all functions of this nature, and on which so many important applications of mathematical analysis are based: *the variations of the demand will be sensibly proportional to the variations in price so long as these last are small fractions of the original price.* Moreover, these variations will be of opposite signs, *i.e.* an increase in price will correspond with a diminution of the demand."

position telle que le léger mouvement de hausse ou de baisse imprimé á la denrée influera sur leur consommation, les engagera à s'imposer quelques privations, ou à rèduire leurs exploitations industrielles, ou à substituer une autre denrée à la denrée renchérie, par exemple, la houille au bois, ou l'anthracite à la houille." [11)]

The Cournot's perceptive recognition in this quotation is that as the market extends wider and the combination of needs, of wealth, and of preferences are varied among consumers, the closer the market demand function $F(p)$ comes to varying continuously with respect to the market price p. One may note, however, that for Cournot the market demand function is rather statistically conceived and is thought to be derived from available market statistical data. Since, in particular, he did not try to derive the market demand function on a purely theoretical ground, he did not go into an explicit discussion about factors having a bearing on the market demand function such as preferences, wealth, etc. of consumers.

2.2 Perception and Representation of Economic Quantity in Walras

Discontinuity of Demand Curve

As did Cournot, Walras also did perceive and very clearly state that " nothing indicates that individual demand curves are continuous; on the contrary they are discontinuous in general, and in reality, their graphs take the form of a step curve ('la forme de la courbe en escalier')" .

> Walras [18, pp.57–58] "...... Rien n'indique que les courbes ou les equations partielles $a_{d,1}a_{p,1}$, $d_a = f_{a,1}(p_a)$ et autres soient *continues*, c'est-à-dire qu'une augmentation infiniment petite de p_a y produise une diminution infiniment petite de d_a. Au contraire,

[11)] Its English tlanslation is as follows: Cournot [4, p.50] "But the wider the market extends, and the more the combinations of needs, of fortunes, or even of caprices, are varied among consumers, the closer the function $F(p)$ will come to varying with p in a continuous manner. However little may be the variation of p, there will be some consumers so placed that the slight rise or fall of the article will affect their consumptions, and will lead them to deprive themselves in some way or to reduce their manufacturing output, or to substitute something else for the article that has grown dearer, as, for instance, coal for wood or anthracite for soft coal."

ces fonctions seront souvent discontinues. Pour ce qui concerne l'avoine, par exemple, il est certain que notre premier porteur de blé réduira sa demande non pas au fur et à mesure de l'elévation du prix, mais d'une facon en quelque sorte intermittente chaque fois qu'il se décidera à avoir un cheval de moins dans son écurie. Sa courbe de demande partielle aura donc en réalité la forme de la courbe en escalier passant au point a (Fig.1). Il en sera de même de tous les autres." [12)]

His explanation of a function $d_a = f_{a,1}(p_a)$ to be continuous ("continue") is that an infinitesimally small ("infiniment petite") increase in the price p_a induces an infinitesimally small decrease in d_a. We may note that Cournot explained the continuity of the demand function without any appeal to the expression of " infinitesimal smallness". [13)]

Immediately following the above quotation, as shown below, Walras claimed that the aggregate or the total demand curve that sums over individual demand curves could possibly be deemed to be continuous in virtue of what he called *the law of large numbers*.

> Walras [18, p.58] "Et cependant, la courbe totale A_dA_p (Fig.2) peut, en vertu de la *loi* dite *des grands nombres*, étre considérée comme sensiblement continue. En effect, lorsqu'il se produira une augmentation trés petite du prix, l'un au moins des porteurs de (B), *sur le grand nombre*, arrivant à la limite qui l'oblige à se priver d'un cheval, il se produira aussi une diminution trés petite de la demande totale." [14)]

[12)] Its English trnslation is as follows: Walras [19, p.169] "There is nothing to indicate that the individual demand curves $a_{d,1}a_{p,1}$ and so on, or the individual demand equations $d_a = f_{a,1}(p_a)$ and so on, are *continuous*, in other words that an infinitesimally small increase in p_a produces an infinitesimally small decrease in d_a. On the contrary, these functions are often discontinuous. In the case of oats, for example, surely our first holder of wheat will not reduce his demand gradually as the price rises, but he will do it in some intermittent way every time he decides to keep one horse less in his stable. His individual demand curve will, in reality, take the *form of a step curve* passing through the point a as in Fig.4. All the other individual demand curves will take the same general form."

[13)] In his book on calculus [6] Cournot explained a differentiable funcion by use of the word of "infinitesimal smallness". I tend to presume that Cournot [4] avoided the use of these words on purpose. However, in case of Walras, I believe he did not intend to claim the differentiability of the demand function in this expression.

[14)] Its English translation is as follows: Walras [19, p.169] "And yet the aggregate demand

Since there is no further explanation as to the law of large numbers to which Walras alluded to, it is not clear at all what he meant exactly by his assertion. The law of large numbers in the sense of the statistics or the probability theory asserts that a sample mean converges under a set of specific conditions to the mean of the population as the size of a sample increases. Thus, it is not proper to interpret what Walras called the law of large numbers in the sense of the statistics or the probability theory. Accordingly a care must be taken to interpret what Walras intended to mean by the law of large numbers.

2.3 Perception and Representation of Economic Quantity in Pareto

Indivisibility of Commodities

With regards to economic quantities, Pareto perceived commodities to be essentially indivisible, and thus he basically considered units of commodities to be measured by integers.

> Pareto [15, p.169] "65. Variazioni continue e variazioni discontinue. — Le curve di indifferenza ed i sentieri potrebbero essere discontinui; anzi nel concreto sono realmente tali, cioè le variazioni delle quantità avvengono in modo discontinuo. Un individuo, dallo stato n cui ha 10 fazzoletti passa ad uno stato in cui ne ha 11, e non già agli stati intermedii, in cui avrebbe per esempio 10 fazzoletti e un centesimo di fazzoletto; 10 fazzoletti e due centesimi, ecc." [15)]

Here the expression of the "continuous variation" (variazioni continue) or rather "discontinuous variation" (variazioni discontinue) refers to indifference

curve A_dA_p (Fig.3) can, for all practical purposes, be considered as continuous by virtue of the so-called *law of large numbers*. In fact, whenever a very small increase in price takes place, at least one of the holders of (B), out of a large number of them, will then reach the point of being compelled to keep one horse less, and thus a very small diminution in the total demand for (A) will result."

[15)] Its English translation is as follows: Pareto [16, p.122] "65. Continuous variations and discontinuous variations. The indifference curves and the paths could be discontinuous, and they are in reality. That is, the variations in the quantities occur in a discontinuous fashion. An individual passes from a state in which he has 10 handkerchiefs to a state in which he has 11, and not through intermediate states in which he would have, for example, 10 and 1/100 handkerchiefs, 10 and 2/100 handkerchiefs, etc."

curves that represent a preference relation in the commodity space. This is in contrast with the previous discussion of Cournot in which case it was the expression on demand functions. Therefore, the Pareto's arguments in the above quotation is not directed to the question of the continuity of demand functions, but is concerned with a question of whether indifference curves can be drawn as continuous curves or not. In other words, we need to interpret his arguments to be based upon the explicit perception of the indivisibility of commodities.

Notwithstanding his fundamental perception of the indivisibility of economic quantities, following the above quotation his arguments proceeded as below:

> Pareto [15, p.169] "Per avvicinarsi al concreto, occorrerebbe dunque considerare variazioni finite, ma c'è una difficoltà tecnica.
>
> I problemi aventi per oggetto quantità che variano per gradi infinitesimi sono molto più facili a trattarsi che i problemi in cui le quantità hanno variazioni finite. Giova dunque, ogni qualvolta ciò si possa fare, sostituire quelli a questi ; e così effettivamente si opera in tutte le scienze fisico naturali. Si sa che per tal modo si fa un errore; ma si può trascurare, sia quando è piccolo in modo assoluto, sia quando è mimore di altri inevitabili, il che rende inutile di ricercàre da una parte una precisione che sfugge dall'altra. *Tale è appunto il caso per l'economia politica, che considera solo fenomeni medii e che si riferiscono a grandi numeri. Discorriamo dell'individuo, non già per ricercare effettivamente cosa un individuo consuma o produce, ma solo per considerare un elemento di una collettività, e per sommare poi consumo e produzione per molti e molti individui.*" [16]

[16] My italics. Its English translation is as follows: Pareto [16, p.123] "In order to come closer to reality, we would have to consider finite variations, but there is a technical difficulty in doing so.

Problems concerning quantities which vary by infinitely small degrees are much easier to solve than problem in which the quantities undergo finite variations. Hence, every time it is possible, we must replace the latter by the former; this is done in all the physiconatural sciences. We know that an error is thereby committed; but it can be neglected either when it is small absolutely, or when it is smaller than other inevitable errors which make it useless to seek a precision which eludes us in other ways. *This is precisely so in political economy, for there we consider only average phenomena and those involving large numbers. We speak of the individual, not in order actually to investigate what one individual consumes*

Thus, Pareto as in the case of Cournot acknowledged that in order to come closer to reality in its analysis one must consider finite variations. Nonetheless, since one faces analytical difficulties in doing so, he proposed to replace quantities going through finite variations ("variazioni finite") with those that vary by infinitesimally small amounts ("gradi infinitesimi") as is done in the natural sciences. Pareto underlined a technical reason at first to treat economic quantities as going through continuous variations.

Economic Quantity as an "Average Phenomenon"

In addition to the pretext of its convenience for analysis and of its conformity with analyses in physical sciences, Pareto in the last part of the above quotation pointed to a more fundamental ground on which such an analysis could be justified. It is this: when the behavior of individual economic agents such as consumers or producers is analyzed, it is an "average phenomenon (fenomeni medi)" that is examined. Furthermore, the number of economic agents is very large ("grandi numeri"). In such a case, even if economic quantities that individual consumers or producers face in reality are finite discrete quantities, a possible error resulting from treating them as quantities varying by "infinitesimally small" amounts ("quantità che variano ser gradi infinitesimi") and hence continuously can be neglected as it is small absolutely or is smaller than other inevitable errors.

Pareto explains what he calls an "average phenomenon" by taking up a concrete example. For example, in the quotation below, he contends that it would be frivolous to take words such as "an individual consumes one and one-tenth watches" literally; rather, it is to be interpreted to signify, for example, that "one hundred individuals consume one hundred and ten watches."

> Pareto [15, p.169] "66. Quando diciamo che un individuo consuma un orologio e un decimo, sarebbe ridicolo il prendere quei termini alla lettera. II decimo dell'orologio è un oggetto sconosciuto e che non ha uso. Ma quei termini esprimono semplicemente che, per esempio, cento individui consumano 110 orologi.
>
> Quando diciamo che l'equilibrio ha luogo quando un individuo consuma un orologio e un decimo, ciò vuol semplicemente es-

or produces, but only to consider one of the elements of a collectivity and then add up the consumption and the production of a large number of individuals."

primere che l'equilibrio ha luogo quando 100 individui consumano chi uno, chi due o più orologi, e anche punti, in modo che tutti insieme ne consumano 110 circa, e che la media per ciaseuno è 1,1. " [17)]

He went on to remind us that this interpretation of individual behavior as an average phenomenon is not limited to economics but it also prevails in other sciences such as in the actuarial science.

> Pareto [15, p.169] "Questo modo non è proprio dell'economia politica, ma appartiene a moltissime scienze. Nelle assicurazioni si discorre di frazioni di viventi; per esempio 27 viventi e 37 centesimi. È pure chiaro che non possono esistere 37 centesimi di un vivente!
>
> Se non si concede di sosotituire le variazioni continue alle discontinue, conviene rinunciare a dare la teoria della leva. Voi mi dite che una leva a braccia eguali, per esempio una bilancia, è in equilibrio quando porta pesi uguali ; io prendo una bilancia che è sensiblile solo al centigramma, metto in uno dei piattini un milligramma di più che nell'altro, e vi faccio vedere che, contraddicendo la teoria, sta in equilibrio.
>
> La bilancia nella quale si pesano i gusti dell'uomo è tale che per alcune merci è sensibile al gramma ; per altre solo all'ettogramma ; per altre solo al chilogramma, ecc.
>
> L'unica conclusione da trarne è che da tali bilancie non bisogna richiedere maggiore precisione di quella che possono dare." [18)]

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17) Its English translation is as follows: Pareto [16, p.123] "66. When we say that an individual consumes one and one-tenth watches, it would be ridiculous to take those words literally. A tenth of a watch is an unknown object for which we have no use. Rather these words simply signify that, for example, one hundred individuals consume 110 watches.

When we say that equilibrium takes place when an individual consumes one and one-tenth watches, we simply mean that equilibrium takes place when 100 individuals consume——some one, others two or more watches and some even none at all——in such a way that all together they consume about 110, and the average is 1.1 for each."

18) Its English translation is as follows: Pareto [16, p.123] "This manner of expression is not peculiar to political economy; it is found in a great number of sciences.

In insurance one speaks of fractions of living persons, for example, twenty-seven and thirty-seven hundredths living persons. It is quite obvious there is no such thing as thirty-seven hundredths of a living person!

2.4 Perception and Representation of Economic Quantity in Marshall

Discontinuity of Individual Demands

In case of Marshall [12] he did acknowledge that there are instances among individual demands for which demands at an individual level do exhibit a behavior representative of the general demand of an entire market such as the demand for tea, and for such individual demands small changes in price induce corresponding small changes in quantities, resulting in a continuous variation. He observed in those instances their demands are constant ones and the commodities can be purchased in small quantities.

However, as the following quotation shows, Marshall too maintained that individual demands in general are discontinuous as did Cournot, Walras, and Pareto. He took as examples watches, hats, etc.

> Marshall [12, p.82] "Section 5. So far we have looked at the demand of a single individual. And in the particular case of such a thing as tea, the demand of a single person is fairly representative of the general demand of a whole market: for the demand for tea is a constant one; and, since it can be purchased in small quantities, every variation in its price is likely to affect the amount which he will buy. But even among those things which are in constant use, there are many for which the demand on the part of any single individual cannot vary continuously with every small change in price, but can move only by great leaps. For instance, a small fall in the price of hats or watches will not affect the action of every one; but it will induce a few persons, who were in doubt whether or not to get a new hat or a new watch, to decide in favour of doing so."

If we did not agree to replace discontinuous variations by continuous variations, the theory of the lever could not be derived. We say that, a lever having equal arms, a balance, for example, is in equilibrium when it is supporting equal weights. But I might take a balance which is sensitive to a centigram, put in one of the trays a milligram more than in the other, and state that, contrary to the theory, it remains in equilibrium.

The balance in which we weigh men's tastes is such that, for certain goods, it is sensitive to the gram, for others only to the hectogram, for others to the kilogram, etc.

The only conclusion that can be drawn is that we must not demand from these balances more precision than they can give."

Continuity of Aggregate Demand in a Large Market

Marshall observed that there are many commodities for which individuals have "inconstant, fitful, and irregular" needs. He argued that for this genre of commodities individual demands are irregular and discontinuous. In spite of the discontinuity of individual demands, he clarified his perception of the continuity of the market demand by lucidly stating that "in large market", "where rich and poor, old and young, men and women, persons of all varieties of tastes, temperaments and occupations are mingled together, the peculiarities in the wants of individuals will compensate one another in a comparatively regular gradation of total demand". This perception of Marshall perfectly matches with that of Cournot excepting that Marshall without doubts had contemplated among individuals a variety of possible factors that induce a regular and continuous variation of the market demand. The following quotation pertains to this. As his arguments are not limited to the one on the continuity of the market demand but also refer to the law of demand, it indicates a probable strong influence of Cournot on Marshall.

> Marshall [12, pp.82-83] "There are many classes of things the need for which on the part of any individual is inconstant, fitful, and irregular. There can be no list of individual demand prices for wedding-cakes, or the services of an expert surgeon. But the economist has little concern with particular incidents in the livers of individuals. He studies rather "the course of action that may be expected under certain conditions from the members of an industrial group," in so far as the motives of that action are measurable by a money price; and in these broad results the variety and the fickleness of individual action are merged in the comparatively regular aggregate of the action of many.
>
> In large markets, then — where rich and poor, old and young, men and women, persons of all varieties of tastes, temperaments and occupations are mingled together, — the peculiarities in the wants of individuals will compensate one another in a comparatively regular gradation of total demand. Every fall, however slight in the price of a commodity in general use, will, other things being equal, increase the total sales of it; just as an unhealthy autumn increases the mortality of a large town, though many persons are uninjured by it. And therefore if we had the

requisite knowledge, we could make a list of prices at which each amount of it could find purchasers in a given place during, say, a year."

3 Perception of Cournot, Walras, Pareto and Marshall in Relation to the Literature in 1950-60's

3.1 Property of Economic Variables: Perception as Quantities or Functions

We have reviewed how representative theorists such as Cournot, Walras, Pareto, and Marshall, each of whom has left significant contributions in the history of economic analysis, perceived and represented economic quantities through their works in the literature. I now wish to summarize their common perception as well as existing dissimilitudes among them.

Preceding the analytical framework of 1950-60's on the general equilibrium analysis as is exemplified in a prototypal work of Debreu [7], we do not seem to encounter explicit discussions concerning how various commodities that are objects of transactions in markets should be quantitatively or mathematically represented. It must be one of the consequences of theoretical efforts toward solving the problem of existence of an equilibrium that such a basic issue comes to light. It thus appears that in the pre-1950's works in the economic analysis, different issues concerning the perception of economic quantities themselves and those concerning the perception of functional forms of economic quantities as economic variables are discussed on the same dimension in a somewhat confused manner.

Common Perception on the Discontinuity of Individual Demand Functions

Cournot [4], in order to analyze mathematically how the values of commodities are determined in market transactions, considered quantities demanded, or synonymously "débit" (sales) in his language, as fundamental economic variables in his analysis. He began his analysis by clarifying demand concept that had been in his time the object of a confusion in the literature. He expressed mathematically the market demand of a commodity as a function of its market price. Since it is the market demand that was on the basis of his analysis, he first discussed its continuity as one of its basic properties.[19)] But, as we saw in Section 2.1, he maintained that demand functions are discontinuous at individual levels. The way he delineated the discontinuity of individual demands are as follows: a slight increase in price does not decrease the quantity demanded slightly in general but as the price increases upto some level, only then the quantity demanded declines abruptly.

The perception that individual demands in general are discontinuous with respect to price variations was commonly held among the theorists Cournot, Walras, Pareto, and Marshall with whose works the present paper is concerned. Despite their common perception we see some essential differences in grounds for their assertion of individual demands being discontinuous. Their different views reflect their varying perception of economic quantities.

Dissimilitudes in Perception of Economic Quantities

One cannot find a trace of Cournot's perception on the indivisibility of economic quantities. He simply observed that individuals in markets do not behave in such a way to adjust their purchases slightly to meet slight price variations.

Walras is on the same boat as Cournot in so far as there were no explicit arguments on the perfect divisibility nor on the indivisibility of commodities. Certainly, Walras knew the content of Cournot [4] when he wrote his book [4]. The point of difference between them is the following: despite the fact that Walras refrained from referring explicitly to the indivisibility of commodities, in my view we should interpret the substance of the quotation from Walras [18, pp.57-58] as a discussion on the basis of admitting the existence of an indivisible commodity (a horse ("cheval") in case of the quotation). As a matter of fact, in case of Cournot, he did not go into a discussion of why individual demands are discontinuous. In case of Walras, however, his argument on this issue seems to stand out as the initiator of the general equilibrium analysis since he essentially argued that even the demand for a *perfectly divisible* commodity (in the example of the quotation, "wheat" (blé)) has the form of a step curve when there is an indivisible commodity.

[19)] As I warned earlier, one must be careful as to whether he actually meant the "continuity" of the market demand or rather its "differentiability".

His reasoning is this: as the price of a perfectly divisible commodity gradually declines, unless it goes down to some level, one might not reduce the consumption of an indivisible commodity by one unit in order to substitute it for an increase of the consumption of a divisible one. In other words his insight of the discontinuous variation of demands is that it accrues to demands from a substitution between a perfectly divisible commodity and an indivisible one as a result of the utility maximizing behavior of an individual induced by a change in the prices of the commodities.

Walras explained that an individual demand function (or rather demand curve) has the form of a step curve. We might be tempted to interpret his explanation literally. But it is highly doubtful whether the true intention of Walras was to maintain that individual demand functions are given by step functions in the present-day mathematical sense.

It is because if we would take such an interpretation, then, they would become semi-continuous functions. The diagram given as Figure 1 in Walras [18, p.58], however, shows a graph of a literal step curve. Thus, we tend to believe that it would be closer to the true intention of Walras to interpret his argument to say individual demands are "correspondences" (or "multi-valued functions") of whose graphs are in the form of a step curve rather than semi-continuous functions whose graphs are in the form of a step curve.

We do not find an explicit discussion of Marshall on the perfect divisibility or the indivisibility of commodities as in the case of his predecessors, Cournot and Walras. But, as Cournot and Walras differ in their perception of the nature of the discontinuity of individual demands, the way Marshall explained it is also at variance in details with them. Cournot avoided to go into a discussion of why individual demands should be regarded as discontinuous. He simply accepted their discontinuity on the basis of his observation of individual behaviors. Both Marshall and Walras based their arguments concerning the discontinuity of individual demands on the preferences of individuals. Marshall, however, explained it from, what we call now, the partial equilibrium analysis point of view, whereas Walras explained it on the basis of a consumption substitution between a perfectly divisible commodity and an indivisible commodity. In his discussion in the first earlier quotation [12, p.82], by taking the consumption of tea as an example, Marshall conceded that individual demand functions to be fairly continuous for those commodities that are consumed constantly and can be purchased by small amounts. Nevertheless, by taking hats and watches as an example, he contended that even among those commodities which are in constant use there are many for which the the demand on the part of any single individual may show a great leap at a certain price level. He does not seem to base his theoretical contention on the observed behavior of individuals, but rather on the individual decision-making. Moreover, in his second earlier quotation [12, pp.82-83], he grounded the discontinuity of individual demands on the irregularity of individual needs that would be observed in cases such as wedding-cakes or the services of an expert surgeon.

In case of Pareto we could see a perspicuous and fundamental difference in his work from that of Cournot, Warlas, and Marshall concerning the divisibility of commodities. As we already pointed it out in Section 2, Pareto developed his arguments with a clear understanding of the indivisible nature of commodities. When Pareto takes up the issue of continuous or discontinuous variations of economic quantities, it is at the level of indifference curves in the commodity space and not at the level of demand functions that are derived from the analysis involving indifference curves in the commodity space. However, Pareto exactly followed the steps of Cournot in emphasizing *sine qua non* to allow for "continuous variations" of economic quantities even at individual levels for technical reasons.

Pareto's arguments might be construed as his way of understanding or extending those of Cournot and Walras since his work was done under the strong intellectual influences of his predecessors.

As far as the issue of the continuity of demands is concerned, the fact that Pareto began his analysis by discussing the continuity or discontinuity of indifference curves which represent individual consumers' tastes and preferences in the commodity space, is regarded as a step toward deepening the understanding of the issue.

3.2 Common Perception on the Continuity of Market Demand

As we reviewed the discussions of Cournot, Walras, Pareto, and Marshall, they all maintain that individual demands are "discontinuous" with respect to changes in prices despite their common understanding that we should nevertheless proceed with our analysis by taking the total market demand to be continuous. Let us classify the rationale for their common understanding in two categories.

One is what Cournot [4, p.39] first emphasized and Marshall [12, pp.82-

83] subsequently elucidated that as wealth, preferences, and needs among individuals constituting the markets extend so as to encompass their wide varieties, total market demand tends to show continuous variations with respect to price changes.

The other is what Walras first referred to as "the law of large numbers" and then Pareto subsequently described as an "average phenomenon." A distinctive feature of Pareto is that he supported his arguments with the understanding of individual economic phenomena as an average phenomenon under large numbers, which gave him a rationale for using continuous real numbers even at individual levels. It might be taken as a way to understand what Walras called in an ambiguous manner the law of large numbers.

We believe it is possible to illuminate the Pareto's insight of interpreting economic quantities as an average phenomenon in a model of "continuum economy" as introduced by Aumann [1] or in a "large economy" as extensively formalized by Hildenbrand [10]. We shall come back to this point in the next section.

4 Perception of Economic Quantity in the History of Economic Analysis and the Debreu Conjecture

4.1 Interpretation of Perception on the "Discontinuity" in the Modern Economic Analysis

Structural Framework of the Commodity Space and Varied Perception on the Discontinuity

In the preceding two sections we reviewed and discussed about the perception of the four representative theorists in the history of economic analysis on the discontinuity of individual demands.

In this section we will present our interpretation of their perception by incorporating the present-day mathematical view point.

As a theoretical structure of the basic commodity space, let us consider two types. One is what we typically find in Dereu [7] and is the standard theoretical model structure of the 1950-60's general equilibrium analysis. The natural number ℓ representing the finite number of distinguishable commodities, the commodity space is given by the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$, and all the commodities are regarded as perfectly divisible. And hence, any real number may represent a physically possible quantity. We shall continue our discussion by taking as an example the simplest type of the commodity space that is the two-dimensional Euclidean space $\mathbb{R}^2$.

The other type of structure of the commodity space is the one where some among ℓ commodities are purely indivisible so that only integer amounts of consumptions are physically possible. In examples with two commodities we shall consider the space

$$\{\cdots\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\cdots\} \times \mathbb{R}$$

as the commodity space instead of $\mathbb{R}^2$. The first commodity is purely indivisible, and the second perfectly divisible allowing for any real number amounts as physically possible consumptions.

In the following we would like to distinguish various types of the perception concerning the "discontinuity" of demand functions. The first type is the "discontinuity" as understood in the present-day mathematics. The second type is the one interpreted as a correspondence or a set-valued (or a multi-valued) function. The third type is the one understood not only as a correspondence but also its failure to be upper hemi-continuous.[20] We may add as the fourth type the nonconvex-valuedness of a correspondence.

Interpretation of the "Discontinuity" in the Commodity Space without Any Explicitly Indivisible Commodities

Walras argued, in his discussion quoted earlier in Section 2.2, that individual demand functions are "discontinuous" taking the form of a step curve. If we make sense of his words literally, he recognized their discontinuity by noticing that they are *correspondences* with their graph having the form of a step curve. By following a Walrasian way of expression, the demand is continuous if an infinitesimally small change in prices induces an infinitesimally small change in the quantity demanded; otherwise, it is discontinuous. As an individual demand which is discontinuous having the form of a step curve, Walras displayed a diagram as in Figure 1. Of course, it is a literal step curve.

[20] The definition or the meaning of the *upper hemi-continuity* is reviewed in a footnote below.

In the commodity space without an explicitly indivisible commodity, let us consider whether one can deduce demand curves having the form that is much the same as the one to which Walras or Cournot referred as discontinuous.

In Figure 2 we tried to exhibit a convex preference relation which induces a step in the graph of its derived demand curve. Taking $\mathbb{R}^2$ as the commodity space, we let the consumption set that represents individual needs and possible individual consumptions be given by the set $\mathbb{R}^2_+ = \{x = (x_1, x_2) | \, x_1 \geqq 0, x_2 \geqq 0\}$. Piecewise linear lines in the figure show typical and representative indifference curves associated with the individual preference relation.

Price vectors $p^A = (p^A_1, p^A_2), p^B = (p^B_1, p^B_2), p^C = (p^C_1, p^C_2)$ correspond to the budget lines BL^A, BL^B, BL^C, respectively. The curve in Figure 3 is the graph of the consumer's demand for commodity 1 which is deduced from the consumption decisions shown in Figure 2 with only the price of commodity 1 changing. The consumer's choice under the budget line BL^A is given by the point x'. In case the budget line is BL^B, the consumptions that the consumer chooses become any one of the points lying along the line segment between (and including) the points x and y. When the budget line further moves to BL^C, the chosen consumptions are given by the points y'.

In this case, the induced demand is not a demand function but is a demand correspondence since all the consumption bundles on the line segment between x and y could be chosen when the price vector is given by p^B. Thus the demand curve has the form of a step but not with multiple steps as in Figure 1. Moreover, one might suspect if Walras (or, for that matter, Cournot) had this type of a step or a jump in mind when either of them talked about the discontinuity of a demand curve.

If one really wants to describe a jump in quantity demanded in some sense within the commodity space without an explicitly indivisible commodity, one might need to appeal to a preference relation not satisfying the convexity. In Figure 4 we exhibit such a preference relation. Piecewise linear lines in the figure show representative indifference curves associated with the individual preference relation. Price vectors $p^A = (p^A_1, p^A_2)$ and $p^B = (p^B_1, p^B_2)$ correspond to the budget lines BL^A and BL^B. The curve in Figure 5 is the graph of the consumer's demand for commodity 1 induced by the consumption decisions shown in Figure 4. Commodity bundles x and y in Figure 4 represent ones that are demanded when market prices are given by the price vector p^A. Hence, in this case also the demand is not a demand function but is given as a demand correspondence. And the quantity of commodity 1 demanded suddenly drops below y_1 if its price rises above the threshold price level p_1^A. Strictly speaking, this example cannot be said to reproduce situations that would fit the description of the discontinuity of individual demands by Walras, Cournot, or Marshall; nevertheless, it might be said to describe circumstances where the quantity demanded of a commodity changes abruptly at a certain level of its price within a framework of the commodity space with only perfectly divisible commodities. [21)]

Interpretation of the "Discontinuity" in the Commodity Space with Explicitly Indivisible Commodities

Our next question is, then, to ask whether one could give an accurate interpretation of the discontinuity that Cournot, Walras, and Marshall perceived by explicitly taking account of indivisible commodities. Let us take as an example the commodity space $\{\cdots\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\cdots\} \times \mathbb{R}$, where commodity 1 is purely indivisible and commodity 2 is perfectly divisible. Then, we take as the consumption set

$$X = \{0, 1, 2, 3, \cdots\cdots\} \times \mathbb{R}_+$$

where possible consumptions consist of commodity bundles with nonnegative consumption amounts. $\mathbb{R}_+$ is the set of all nonnegative real numbers. Figure 6 exhibits an example of a consumer's behavior with X as its consumption set. In the figure the consumption set consists of perpendicular half-lines. The preference relation of the consumer is representatively shown by several indifference curves that are given by the consumption vectors lying on both the dotted curves and the perpendicular half-lines.

Let the price vectors $p^A = (p_1^A, p_2^A), p^B = (p_1^B, p_2^B), p^C = (p_1^C, p_2^C)$ correspond to the budget lines BL^A, BL^B, BL^C, respectively. The consumer's choice under the budget line BL^A is shown by the point x. Under the budget line BL^B, the consumer chooses the points x' and y. When the budget line is BL^C, the chosen consumptions are given by the points y' and z. This consumer's choice behavior in Figure 6 with respect to the quantity of commodity 1 demanded is shown in Figure 7 as the curve of its individual demand function. The figure does not exactly matches a piecewise linear step curve

[21)] Note, however, that here the demands as a correspondence satisfy the property of upper hemi-continuity. It simply represents the lack of convex-valuedness of the correspondence at the price vector p^A.

that Walras drew as in Figure 1 but we believe it is possible to regard it to represent what Walras [18, p.57] explained with respect to the variations of individual demands.[22] Moreover, we might be allowed to say that it also represents the discontinuity of individual demands discussed by Cournot [4, p.38-39] and Marshall [12, p.82].

Nevertheless, one might wonder whether the perception of Cournot on the discontinuity of individual demand functions, as compared to that of Walras, Pareto or Marshall, in a sense lies on a deeper ground. Should it be so, how would we understand his perception on the discontinuity of the demands?

Considering the fact that Cournot was a mathematician before he was an economic theorist and that he had written among others textbooks on calculus (Cournot [6]) and on probability theory, it would not be appropriate to think that he regarded individual demands to be functions having the form of a step curve as in the case of Walras. I believe we could interpret the earlier quotation of Cournot [4, p.38-39] to mean that an individual demand for a particular commodity as a real-valued function is at most a *semi-continuous* function.[23]

It is difficult to infer whether the Cournot's perception on the discontinuity of individual demand functions was based upon a further deeper ground or not. Let us be more specific about this point using Figure 8 and Figure 9. In Figure 8 each of the budget lines BL^A, BL^B, BL^C, BL^D, corresponds to the

[22] However, if we take the explanation of Walras [18, p.57] to indicate changes in quantity demanded of a perfectly divisible commodity resulting from its substitution for an indivisible one, then these figures fail to represent his explanation. The Walras' explication might seem persuasive at first sight, but if we depict the consumer's choice in Figure 6 by the demand function for commodity 2, then, strictly speaking, it seems to us that his insight might have been misled.

[23] In particular, I would like to call the attention of readers to the fact that his explanation in the quotation could be understood to mean that an individual demand function as a real-valued function cannot be (lower semi-) continuous even though it might be upper semi-continuous.

A real-valued function $f : X \to \mathbb{R}$ is *upper semi-continuous* at $x \in X$ if the set $\{z | f(z) < f(x)\}$ is open, and f is *upper semi-continuous* if it is upper semi-continuous at all $x \in X$.

$f : X \to \mathbb{R}$ is *lower semi-continuous* at $x \in X$ if the set $\{z | f(z) > f(x)\}$ is open, and f is *lower semi-continuous* if it is lower semi-continuous at all $x \in X$.

f is said to be *semi-continuous* if it is either upper semi-continuous or lower semi-continuous at all $x \in X$.

Note that even if a real-valued function is upper semi-continuous, it need not be upper hemi-continuous when it is regarded as a correspondence. See a footnote below for the concepts of the upper hemi-continuity and the lower hemi-continuity of a correspondence.

price vector $p^A = (p_1^A, p_2^A), p^B = (p_1^B, p_2^B), p^C = (p_1^C, p_2^C)$, or $p^D = (p_1^D, p_D)$, respectively. Using the price-consumption curve in Figure 8, these figures show how an individual demand changes responding to the changes of the price of commodity 1 from p_1^A to p_1^D. The demand curve of commodity 1 derived from this price-consumption curve is the graph of the correspondence having the "form of a step curve" in Figure 9. Blackened points indicate they are a part of the curve whereas simply circled points are not a part of the curve. The correspondence in Figure 9 is a function except at the price vector p_1^D. Within the region where it becomes a function, it is semi-continuous as a real-valued function. It is *not lower semi-continuous* but is upper semi-continuous. Regarded as a correspondence in the whole region containg the price vector p_1^D, the demand is not convex-valued at the price p_1^D but is continuous there, i.e., it is upper hemi-continuous as well as lower hemi-continuous at p_1^D. However, it is *not upper hemi-continuous* at p_1^B and p_1^C.[24)]

It is not clear to the present author whether the Cournot's awareness of the discontinuity of individual demands was about their nonconvex-valuedness as exemplified in Figure 6 and Figure 7 or more insightfully about their lack of upper hemi-continuity. The demand correspondence in Figure 9 is not convex-valued at p_1^D. If Cournot implicated such a situation by [4, pp.38-39] as a lower semi-continuous step function as in our earlier footnote, we should admit that he did not apperceive the lack of upper hemi-continuity at p_1^B, p_1^C as a demand correspondence. Since the correspondence in Figure 9 is single-valued at p_1^B, p_1^C, it is upper semi-continuous at these prices when viewed as a function. One could say that Cournot [4, pp.38-39] simply meant that demand functions are semi-continuous without further perceptive distinction of upper or lower semi-continuity. In that case it would be possible to say that he had an apprehension of the fact that the demands as a correspondence might fail to be upper hemi-continuous.[25)]

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24) A correspondence $F : X \to Y$ is *upper hemi-continuous* at $x \in X$ if for any open set $G \supset F(x)$ in Y, there exists an open set V with $x \in V$ such that for every $z \in V$ one has $F(z) \subset G$. F is *upper hemi-continuous* if it is upper hemi-continuous at every $x \in X$.

F is *lower hemi-continuous* at $x \in X$ if for any open ste G in Y with $F(x) \cap G \neq \emptyset$, there exists an open set V with $x \in V$ such that for any $z \in V$ one has $F(z) \cap G \neq \emptyset$. If F is lower hemi-continuous at every $x \in X$, then F is *lower hemi-continuous.*

25) Since the proofs of the existence of an equilibrium in the general equilibrium model in 1950-60's were carried out in a framework where individual demand correspondences essentially become upper hemi-continuous, the awareness of circumstances under which individual demand correspondences fail to be upper hemi-continuous seems to be fairly

To sum up our views on how Cournot, Walras, and Marshall perceived the discontinuity of individual demands and how Pareto perceived the discontinuity:

(1) Except for Pareto, there are no explicit arguments involving indivisible commodities. And we do not obtain a demand function with its graph having the "form of a step curve" that they referred to in their words or in their diagrams within a framework of the commodity space with perfectly divisible commodities only.

(2) By incorporating an indivisible commodity into the commodity space explicitly, one could obtain a demand curve of the "form of a step curve" by interpreting the fact that the demand correspondence is not convex-valued to implicate a situation representing a "step" in a weak sense.

(3) Notwithstanding an interpretation of Cournot, Walras, and Marshall to the effect that they took account of indivisible commodities, their arguments do not lead to the recognition of the lack of upper hemi-continuity of a demand correspondence.

(4) In case of Cournot there are some grounds left to believe that his arguments could be taken to imply his perception of the lack of the upper hemi-continuity of individual demand correspondences when some of the commodities are indivisible.

(5) In case of Pareto his argments recognized indivisible commodities in the commoditiy space in a straightforward way, and perceived the discontinuity of indifference curves upon which the derivation of a demand curve depends.

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limited.

As we typically see in Debreu [7, 4.8, p.63], an individual demand correspondence may fail to be upper hemi-continuous if the level of wealth of that individual drops down to the minimal level so as to sustain the purchase of the least expensive combinations of commodities among all the possible consumptions. Thus, in most of the cases, existence proofs have been carried out under conditions where all the economic agents circumvent the situation of the minimum level of their wealth for possible consumptions.

Now, in Figure 7, the circumstances under p_1^B or p_1^C do not correspond to the "minimum wealth level" among all the possible consumptions. But they arose from the existence of an indivisible commodity. In the literature these circumstances were clarified by Broome [3], Mas-Colell [14], and Yamazaki [20], [22].

4.2 The Debreu Conjecture and Interpretation of the Common Perception on the Continuity of the Market Demands

We now wish to go back to the Debreu conjecture in this final section of the paper and attempt to propose a present-day interpretation of the common perception on the continuity of the market demands.

As we reviewed in the previous section, each of Cournot, Walras, Pareto, and Marshall takes individual demands to vary discontinuously with respect to the changes in prices, but they commonly perceives that the total market demand varies continuously with respect to price changes.

If we review the content of the Debreu conjecture, it goes as "One expects that if the measure ν is suitably diffused over the space A (of economic agents' characteristics), integration over A of the demand *correspondences* of the agents will yield a total *demand function*, possibly even a total demand function of class C^1." The first half of this conjecture reflects in modern mathematical terms what Cournot [4, p38] had stated as "the wider the market extends, and the more the combination of needs, of fortunes, or even caprices, are varied among consumers", and what Marshall [12, p82-83] had stated as "large markets ... where rich and poor, old and young, men and women, persons of all varieties of tastes, temperaments and occupations are mingled together ...... the peculiarities in the wants of individuals will compensate one another in a comparatively regular gradation of total demand." Debreu did not mention Cournot nor Marshall at all in his paper [12] when he promulgated his conjecture. I personally find it very hard to believe that Debreu did not have any knowledge about the Cournot's idea on this issue.

The last half of the conjecture consists of two parts: one part is to assert that the total market demand obtained from aggregation over individual demands will possibly become a function, i.e., a single-valued correspondence, regardless of whether individual demands are correspondences or not provided that there are enough "diffusion" or variation of wealth levels, preference relations, needs, etc., among individual agents; the other part is to claim the possibility of having a continuously differentiable function as a total market demand once it becomes a function.

The first part, in my view, is definitely a literal translation of the assertion by Cournot [4, pp.38-39] and the subsequent one by Marshall [12, pp.82-83] in a large economy representation of the general equilibrium analysis in as much as the fact that the total demand is a function within the framework of 1950-60's implies that demands change continuously with respect to price changes.[26)]

As to the second part of the last half of the conjecture, it might seem on the surface that no one among Cournot, Walras, Pareto, and Marshall referred to it. But, as we pointed it out in Section 2, we suspect that, from the Cournot's description about the property of a continuous function, when he discussed the *continuity* of demand functions, he might have had in his mind the *differentiability* of demand functions instead under the name of their continuity.[27)] Indulging ourselves in such an interpretation, the argument of Cournot [4, pp.38-39] becomes essentially the Debreu conjecture itself. In other words, by reversing the way of its statement, we could say that the Debreu conjecture succeeded in giving a formal statement of the Cournot's idea in terms of the present-day economic theory.

Next, aside from our discussion on the direct significance of the conjecture, we believe it appropriate to discuss about a possible common lineage between the conjecture and the perception of "the law of large numbers" as Walras alluded to, or "the average phenomenon" as Pareto pointed out.

Take a continuum economy as introduced by Aumann [1]. And, let $I = [0, 1]$ be an index set of the population of individuals composing an economy and λ be the Lebesgue measure on I with $\lambda(S)$ representing the proportion of individuals belonging to a group S of the people in I. For each $t \in I$, $F(t, p)$ is the value of individual demand correspondences, and it shows the set of demand vectors of individual agent t under the price vector p. In a continuum economy the integral $\int_I F(t, p) d\lambda$ represents the set showing the value of the aggregate total demand correspondence at p.[28)]

Hildenbrand [10], who systematically expanded the framework of a con-

26) It may be better to give a remark on a question whether a demand correspondence being a function implies de facto it being continuous. Within the framework of the Debreu conjecture all the commodities are perfectly divisible, and in his general equilibrium model with a differentiable structure of preference relations, as we noted in our earlier footnote, circumstances under which individual demand correspondences may fail to be upper hemi-continuous are excluded so that the mere fact of being demand *functions* guarantees the continuity of demand functions.

27) For, as we gave a remark in an earlier footnote, he pointed out, as a characteristics of a continuous function, the property of its local linearity. What's more, he described this property as a property of a differentiable function in his textbook on calculus [6, pp.9-10].

28) For the concept of integrals in such a mathematical model, please refer to a book by Hildenbrand [10], Jacobs [11], Maruyama [13], or Yamazaki [**?**].

tinuum economy to a model of large economies, called an aggregate total demand as a *"mean demand"*. It is because units of the aggregate total demand $\int_I F(t,p)d\lambda$ are interpreted to be amounts expressed in terms of units per capita among the population of economic agents in an economy. Now, the value of mean demand $\int_I F(t,p)d\lambda$ as the aggregate total demand is known to be convex-valued in a continuum economy.[29)]

Walras, in discussing about the market total demand by aggregating over individual demands, made reference to "the law of large numbers" without giving any further detailed comments or explanation. Thus, we believe it reasonable to surmise what he meant to be is the fact that mean demand $\int_I F(t,p)d\lambda$ becomes convex-valued. In other words, the law of large numbers in the sense of Walras is understood to be a phenomenon of straightforward convexifying effects inherent to a process of aggregation itself over a large number of individual demands.

How about, then, what Pareto called "the average phenomenon" of economic quantities? Clearly, one might wish to regard the mean demand representing an aggregate total demand as "the average phenomenon" of economic quantities. However, this forthright interpretation does not seem to be a precise representation of the average phenomenon of Pareto in view of the fact that he was concerned about the treatment of quantities in the commodity space. He argued that even though each individual faces in reality choices among discrete amounts, theoretically we are allowed to regard each individual to make decisions among continuous economic quantities since we are only interested in an average phenomenon of those individuals.

For the purpose of articulating the Pareto's arguments, let us take up the commodity space where some of the commodities are explicitly indivisible. Since quantities are confined to discrete amounts, the commodity space would become nonconvex, and so are individual preference relations. Thus, as an analytical convenience, consider the convexification of consumption sets and preference relations. In this procedure of convexification we can allow individuals to make choices among consumption bundles composed of continuous real numbers not confined to discrete amounts. Then, if the values of the actual mean demand arising from consumption sets and preference relations without their convexification, can be shown to coincide with those of the mean demand resulting from the convexified consumption sets and pref-

[29)] This is a concequence of a well-known mathematical theorem due to Lyapunov. See, for example, Hildenbrand [10, Theorem 3, p.62] or Yamazaki [**?**, Theorem 14.2, p.186].

erence relations, then this procedure of convexification can be understood to represent the Pareto's idea.[30)]

We are more than willing to concede that the representation as well as the perception of quantities and varialbes in the economic analysis clearly has a correlation with the development of the mathematical analysis. Hence, as our future research, we would like to deepen our understanding of how the representation of quantities and their variables found in the history of the mathematical analysis might be understood to be related to the perception of economic quantities.

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[30)] If one is interested in seeing to what extent such a proposition might be shown to be true, see, e.g., Yamazaki [**?**, Theorem 14.1 or 14.2, pp.184-187].

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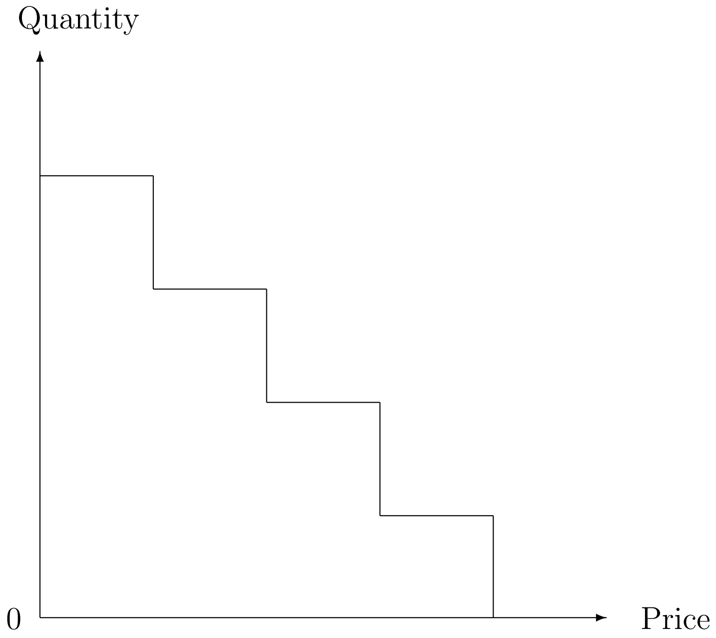

Figure 1: An Individual Demand Curve

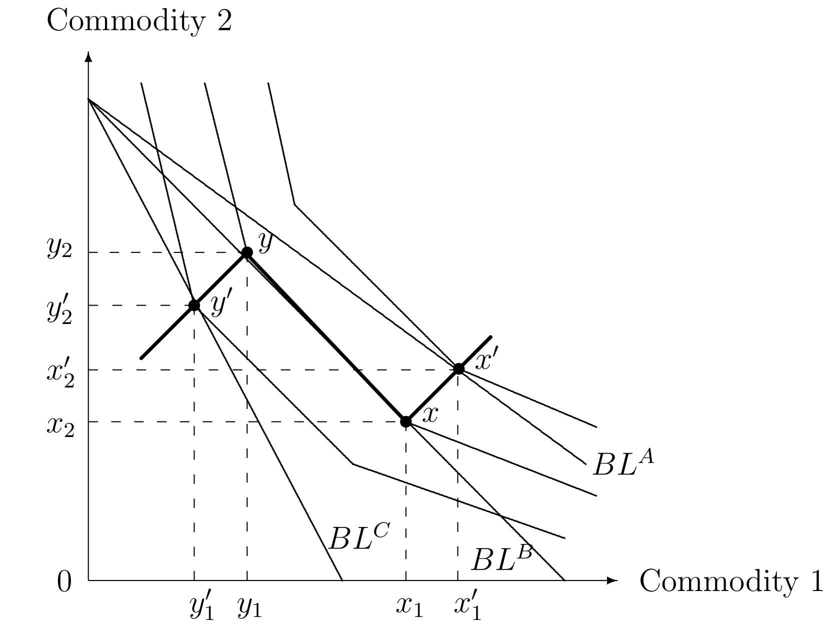

Figure 2: Consumer's Choice

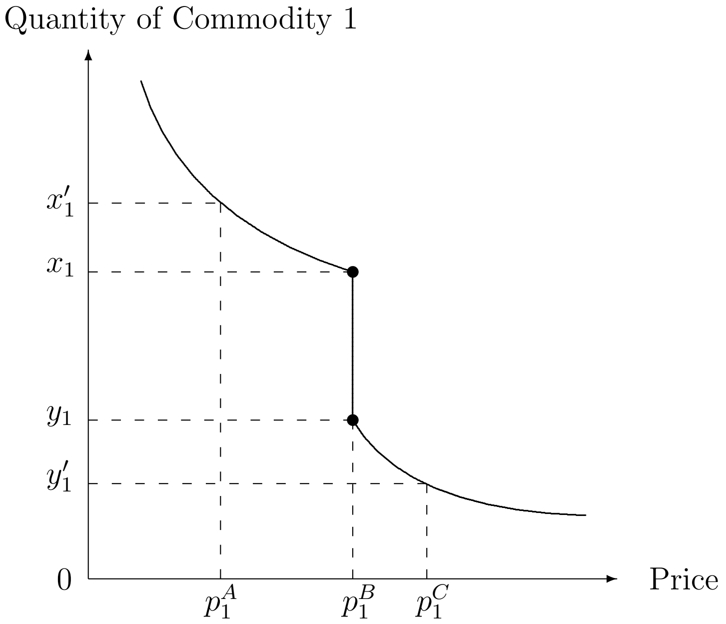

Figure 3: Consumer's Demand Curve for Commodity 1

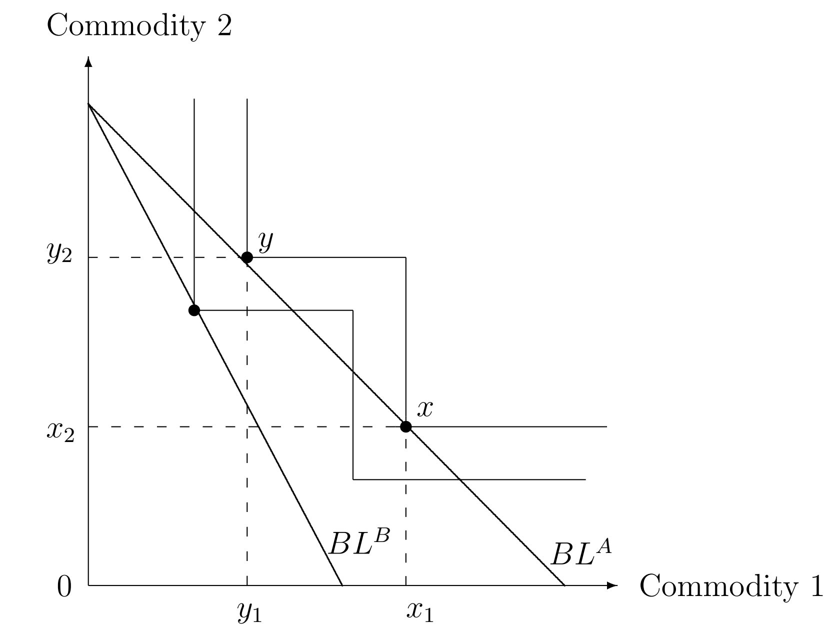

Figure 4: Consumer's Choice

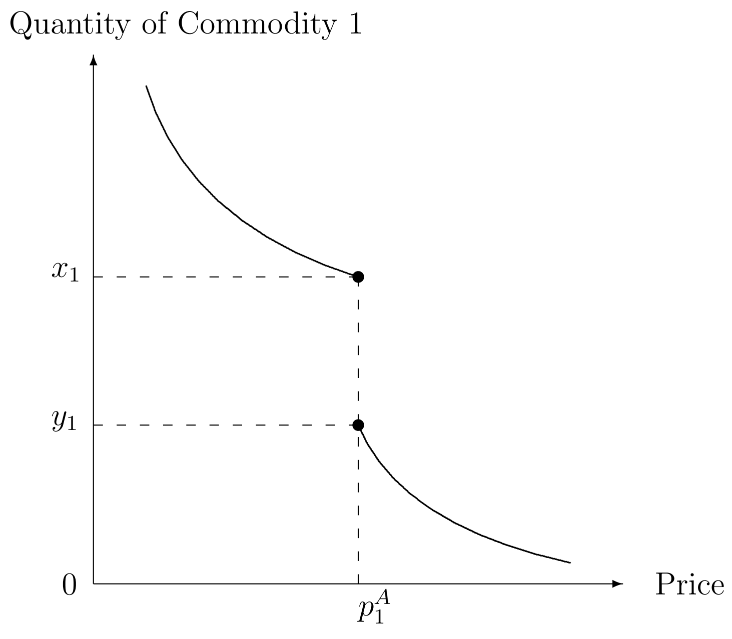

Figure 5: Consumer's Demand Curve for Commodity 1

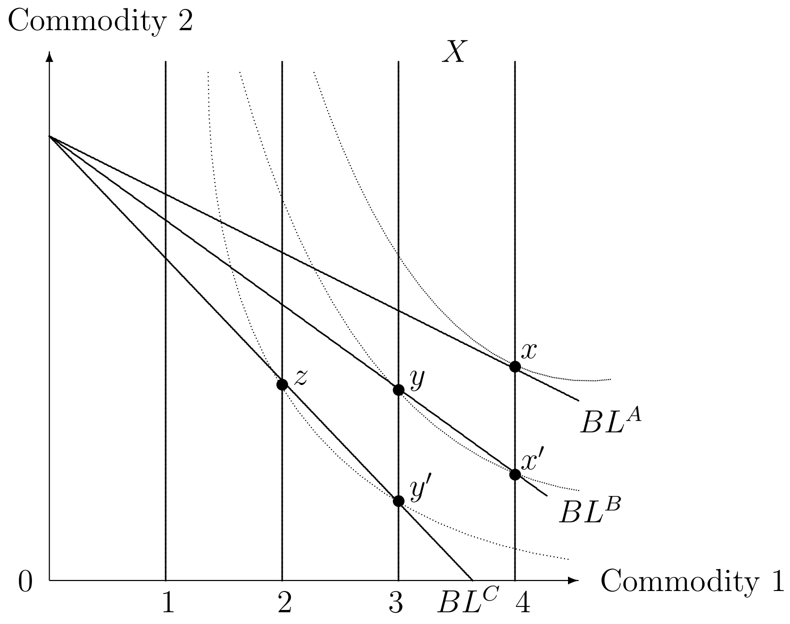

Figure 6: Consumer's Choice

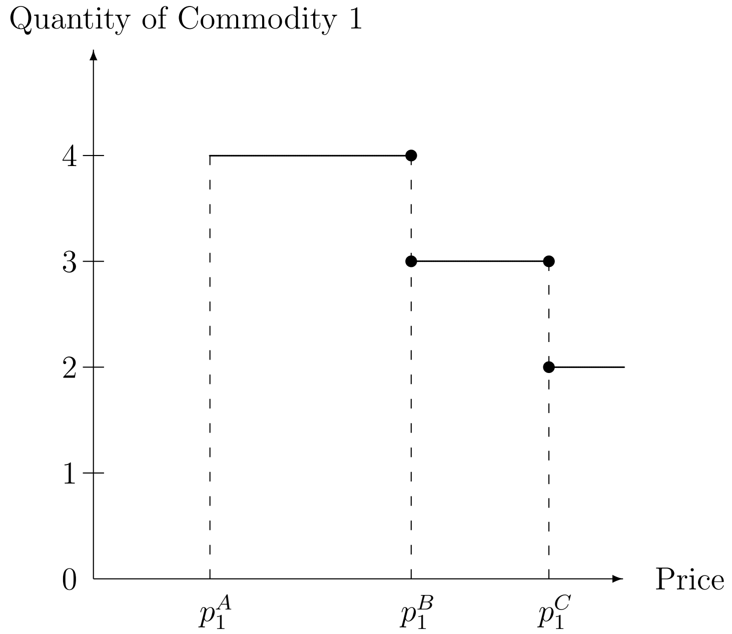

Figure 7: Consumer's Demand Curve for Commodity 1

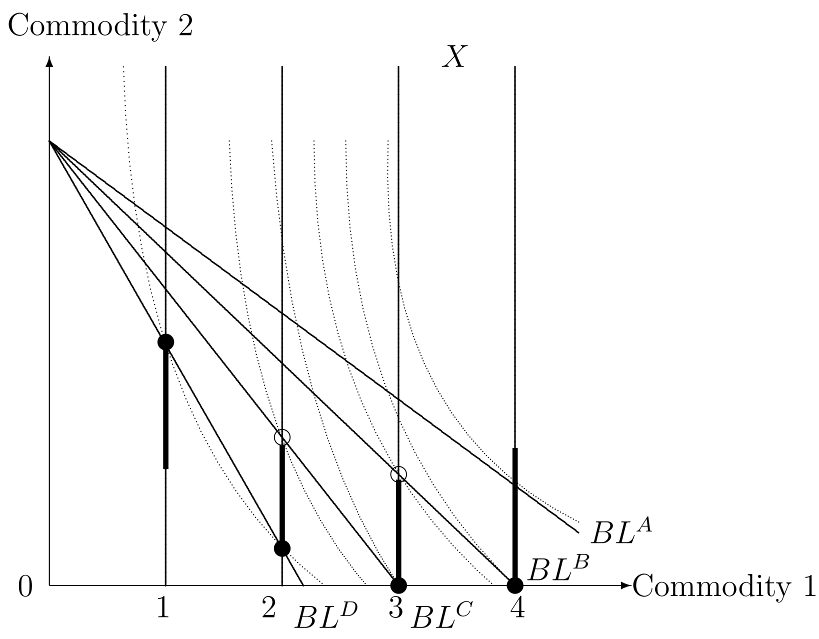

Figure 8: Consumer's Choice

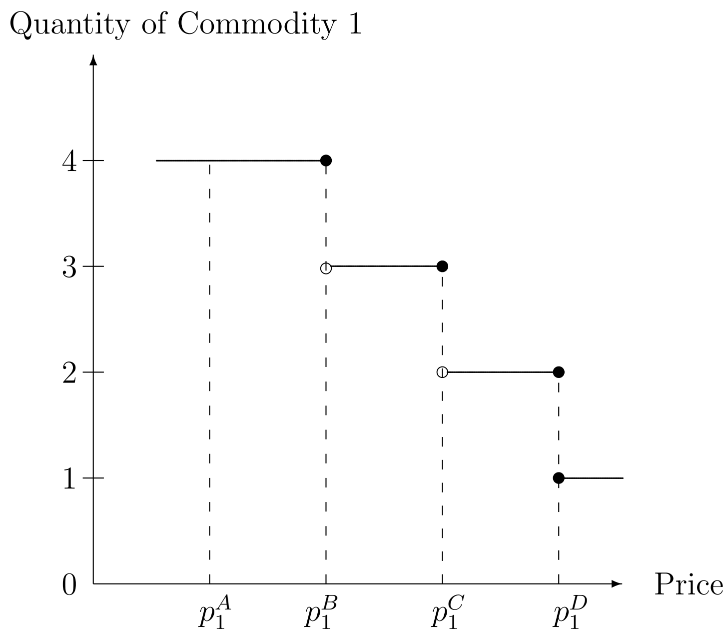

Figure 9: Consumer's Demand Curve for Commodity 1

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