Discussion Paper Series

Graduate School of Economics and School of Economics

Meisei University

Discussion Paper Series, No. 32

May, 2015

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Duality in Nonlinear Pricing with Applications to Block Tariffs

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March 7, 2015

Abstract

The aim of the paper is to establish a method for constructing a price schedule as an indirect mechanism from any incentive compatible and individually rational direct revelation mechanism in a quasi-linear context. The paper presents a necessary and sufficient condition for the optimality of posting a block tariff.

Keywords: nonlinear pricing · taxation principle · bunching · block tariffs · countervailing incentives

JEL Classification Numbers: D4 · D11 · D82 · D86

1 Introduction

I consider a model of contracting by a principal and an agent under asymmetric information. There is adverse selection because the agent's characteristics/types are not observable to the principal (the distribution being known, however). The principal and the agent contract on a product characteristic such as quality and quantity, and a monetary transfer. The agent's product choice can be described as a *decision rule* that assigns a product choice for each type. The strategy space of the principal is a set of nonlinear price schedules. Each nonlinear pricing schedule is considered a catalog of products and prices. The agent chooses a product so as to maximize his utility from trade. I say that a decision rule is *implementable via price schedule* if it is consistent with the agent's

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type-dependence best responses for some price schedule. The principal selects a price schedule so as to maximize his expected profit from trade, subject to the implementability constraints and the participation constraints.

The standard approach to the screening problem is reformulating the principal's strategy space. If a decision rule is implementable by a nonlinear price schedule, then a transfer function indexed by agent types is obtained as the composite function of the price schedule and the decision rule. By construction, such a direct revelation mechanism is incentive compatible in the sense that it is optimal for the agent to announce the true value of his private information. This is so-called the *revelation principle*.[1] The principal's problem is rewritten as an expected profit maximization problem over a set of incentive compatible decisions and transfers satisfying the participation constraints.

The use of the revelation principle has become widespread in the optimal contracts literature. However, the literature has focused on analyzing the properties of contracts indexed by agent types, such as distortions and information rents, rather than the properties of price schedules. The purpose of the paper is to establish the reverse of the revelation principle, that is, to construct an indirect mechanism or a price schedule as a solution to the principal's optimization problem.

It is not difficult to construct a nonlinear price schedule implementing a given decision rule when the decision rule exhibits no pooling (cf. Laffont and Martimort, 2002, Proposition 9.6). My concern in Section 3 is how to construct such a price schedule in a more general setting in which the reservation utility is type-specific and bunching may arise. Maggi and Rodriguez-Clare (1995) argue that when the reservation utility depends on the private information, the structure of optimal contracts, in particular, the occurrence of pooling, crucially depends on the shape of the reservation utility function. Theorem 2 in the paper establishes the implementability of any decision rules possibly involving bunching in a general model of nonlinear pricing. I introduce the notion of *voluntary implementability*, taking into account the agent's type-dependent reservation utility. Theorem 2, together with the revelation principle, states that any pair of a decision rule and an information rent is incentive compatible and individually rational if and only if it is voluntarily implementable by some indirect mechanism (Theorem 3). The advantage of my construction is that there is no need to exclude bunching in decision rules. There is a procedure to obtain a nonlinear price function from any incentive compatible and individually rational direct revelation mechanism. Then, it means to claim that, without loss of generality, we can restrict attention to a class of direct revelation mechanisms.

In Section 4, I explore the economic interpretations behind the constructed price schedule in the previous section. The type-assignment approach examined by Goldman et al. (1984) and subsequent analysis by Nöldeke and Samuelson (2007) solves screening problems by using the inverse of a decision rule instead of a decision rule. It is shown that the method taken in the paper is closely related with the type-assignment approach (Theorem 5). In addition, Corollary 1 states

[1] For instance, see Laffont and Martimort (2002, Proposition 2.2).

that the paper connects the two approaches based on direct revelation mechanisms and indirect mechanisms.

In Section 5.1, I discuss under what conditions, the optimal price schedule belongs to the principal's strategy space consisting of *piecewise linear* price schedules. Piecewise linear tariffs are commonly used to public utilities. For instance, TEPCO (Tokyo Electric Power Company, Inc) uses increasing block tariffs as shown in Figure 1. On the other hand, Tokyo Gas Co., Ltd. uses decreasing block tariffs. I show that a single equation is necessary and sufficient for that the optimal price schedule constructed in the proof of Theorem 2 takes the form of a block tariff (Theorem 6). In Section 5.2, I examine the character of nonlinear pricing including convexity/concavity with respect to quality/quantity remains largely unexplored because of the use of direct revelation mechanisms in the literature. The expression for the second-derivative of the optimal price schedule can be used to check the optimality of quality premia or quantity discounts (Proposition 4).

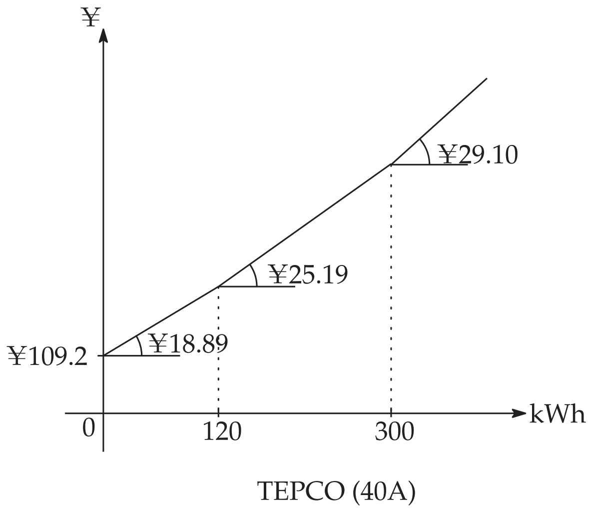

Figure 1: Increasing Block Tariff

2 Principal-Agent Model in a Quasi-Linear Context

I consider a principal-agent model which can be described as follows. The principal is a Stackelberg leader of the two-player game with asymmetric information. In the first stage, the principal chooses his strategy given the optimizing behavior of the agent in the second stage. The principal and the agent contract on two types of variables: a product characteristic x such as quality or quantity, and a monetary transfer y. Both are observable to both players. Denote by $X \subseteq \mathbb{R}_+$ the product line that the principal can offer. A strategy of the principal is a nonlinear *price schedule* $t : X \to \mathbb{R}$. The principal's utility is $y - C(x)$, where $C : X \to \mathbb{R}$ is the cost function.

The agent chooses a product x sold at price y. Ignoring income effects, his choice is made according to preferences represented as a quasi-linear utility function $u(x, \theta) - y$, where type θ

is a one-dimensional parameter that belongs to a compact set $\Theta = [\underline{\theta}, \overline{\theta}] \subseteq \mathbb{R}_{++}$.[2] There is adverse selection problem because this parameter is known to the agent but unobservable to the principal. I assume that $u_x(x,\theta) > 0$ and $u_{x\theta}(x,\theta) > 0$.[3] The latter condition is called the single-crossing property. Given a price schedule $t(\cdot)$, a consumer of type θ maximizes his net utility $u(x,\theta) - t(x)$ over X. Let $\pi_t(\theta) = \max[u(x,\theta) - t(x) \mid x \in X]$ be the indirect utility function of type θ. Finally, the agent of type θ may have an outside opportunity, from which he can derive a utility level $\bar{\pi}(\theta)$.

From the principal's perspective, the agent's type is continuously distributed over Θ with a density function $f(\theta) > 0$ for every $\theta \in \Theta$. The principal chooses a price schedule to solve the following profit-maximization problem:

$$\max_{t(\cdot)} \int_{\Theta} [t(x(\theta)) - C(x(\theta))] f(\theta) d\theta$$

subject to the *implementability constraints*

$$x(\theta) \in \operatorname{argmax}[u(x,\theta) - t(x) \mid x \in X], \quad \forall \theta \in \Theta$$

and the *participation constraints*

$$\pi_t(\theta) = \max[u(x,\theta) - t(x) \mid x \in X] \geqslant \bar{\pi}(\theta), \quad \forall \theta \in \Theta.$$

The standard approach to the screening problem is reformulating the principal's strategy space. The participation constraints can be replaced by the system of inequalities, $r(\theta) = \pi_t(\theta) - \bar{\pi}(\theta) \geqslant 0$ for every $\theta \in \Theta$. Introducing the social surplus function $v(x,\theta) = u(x,\theta) - C(x)$, I rewrite profit margin as $t(x(\theta)) - C(x(\theta)) = u(x(\theta),\theta) - \pi_t(\theta) - C(x(\theta)) = v(x(\theta),\theta) - r(\theta) - \bar{\pi}(\theta)$. Moreover, if $x(\theta) \in \operatorname{argmax}[u(x,\theta) - t(x) \mid x \in X]$ holds under a price schedule $t(\cdot)$, then it is the case $\theta \in \operatorname{argmax}[u(x(\hat{\theta}),\theta) - t(x(\hat{\theta})) \mid \hat{\theta} \in \Theta]$. Using the information rent, the payment $t(x(\hat{\theta}))$ in this expression is written as $t(x(\hat{\theta})) = u(x(\hat{\theta}),\hat{\theta}) - \pi_t(\hat{\theta}) = u(x(\hat{\theta}),\hat{\theta}) - r(\hat{\theta}) - \bar{\pi}(\hat{\theta})$. To sum up, a profile $\langle x(\cdot), r(\cdot) \rangle$ of a decision rule and an information rent is *incentive compatible* in the sense that $\theta \in \operatorname{argmax}[u(x(\hat{\theta}),\theta) - u(x(\hat{\theta}),\hat{\theta}) + r(\hat{\theta}) + \bar{\pi}(\hat{\theta}) \mid \hat{\theta} \in \Theta]$ for every $\theta \in \Theta$. In what follows, such a profile $\langle x(\cdot), r(\cdot) \rangle$ is called a *direct revelation mechanism*.

[2] The literature generally assumes that there are no significant income effects. There are some exceptions. Wilson (1993, Chapter 7) discusses how income effects can influence the design of nonlinear pricings for the case of a profit-maximizing monopolist. Goldman et al. (1984, pp.315-316) and Roberts (1979) study Ramsey pricing in the context of public utilities pricing when consumer's type is the same as his income. Watabe (2015) studies the design of nonlinear pricing schedules when consumers' preferences are formulated as in Goldman et al. (1984, pp.315-316) and Roberts (1979).

[3] Throughout the paper, subscripts denote partial derivatives of the agent's utility function $u(x,\theta)$. Moreover, I indicate derivatives taken with respect to θ with a "dot" superscript, while derivatives with respect to x with "prime" superscript.

The principal's problem can be written as

$$\max_{\langle x(\cdot), r(\cdot)\rangle} \int_{\Theta} [v(x(\theta),\theta) - r(\theta) - \bar{\pi}(\theta)] f(\theta) d\theta$$

subject to the incentive constraints

$$\theta \in \operatorname{argmax}[u(x(\hat{\theta}),\theta) - u(x(\hat{\theta}),\hat{\theta}) + r(\hat{\theta}) + \bar{\pi}(\hat{\theta}) \mid \hat{\theta} \in \Theta], \quad \forall \theta \in \Theta,$$

and the participation constraints

$$r(\theta) \geqslant 0, \quad \forall \theta \in \Theta.$$

Finally, a direct revelation mechanism $\langle x(\cdot), r(\cdot)\rangle$ is *individually rational* if $r(\theta) \geqslant 0$ for every $\theta \in \Theta$. The monotonicity of the information rent $r(\cdot)$ is not guaranteed in general.

3 Reconsideration of Rationalizability in Rochet (1987)

In this section, I construct a price schedule under which any non-decreasing decision rule $x : \Theta \to X$ emerges as a solution to utility maximization problem of consumers. Without loss of generality, I may assume that $X = [x(\underline{\theta}), x(\overline{\theta})]$, given a non-decreasing decision rule $x(\cdot)$.

The literature has been focused on the following incentive compatibility through a transfer function defined over the type space.

Definition 1 (Rochet, 1987). A decision rule $x(\cdot)$ is said to be *rationalizable or implementable via transfer* if there exists a *transfer function* $p : \Theta \to \mathbb{R}$ such that the direct revelation mechanism $\langle x(\cdot), p(\cdot)\rangle$ induces truthful revelation: $\theta \in \operatorname{argmax}[u(x(\hat{\theta}),\theta) - p(\hat{\theta}) \mid \hat{\theta} \in \Theta]$ for every $\theta \in \Theta$.

On the other hand, my concern is the implementability in the following sense.

Definition 2. A decision rule $x(\cdot)$ is said to be *implementable via price schedule* if there exists a price schedule $t : X \to \mathbb{R}$ such that $x(\theta) \in \operatorname{argmax}[u(x,\theta) - t(x) \mid x \in X]$ for every $\theta \in \Theta$.

When the reservation utility is not type-dependent, Rochet (1985, Principle 2) shows how to recover the requirement (1) in the above definition. The participation constraints are incorporated into the implementability in the following manner.

Definition 3. A direct revelation mechanism $\langle x(\cdot), r(\cdot)\rangle$ is *voluntarily implementable via price schedule* if there exists a price schedule $t : X \to \mathbb{R}$ such that for every $\theta \in \Theta$,
(1) $x(\theta) \in \operatorname{argmax}[u(x,\theta) - t(x) \mid x \in X]$,
(2) $r(\theta) = u(x(\theta),\theta) - t(x(\theta)) - \bar{\pi}(\theta) \geqslant 0$.

The necessary conditions for the voluntary implementability are summarized as follows.

Theorem 1 (Revelation Principle). *If a direct revelation mechanism $\langle x(\cdot), r(\cdot)\rangle$ is voluntarily implementable, then it is incentive compatible and individually rational.*

Proof. The revelation principle (cf. Laffont and Martimort, 2002, Proposition 2.2) states that condition (1) in Definition 3 implies the incentive compatibility. The individual rationality is trivially satisfied by condition (2) in Definition 3. This establishes the theorem. □

In what follows, I shall explore the reverse of the revelation principle. My question is whether it is possible to construct a nonlinear price schedule for *any* given incentive compatible and individually rational direct revelation mechanism for the voluntary implementability.

Assumption 1. The reservation utility function $\bar{\pi}(\cdot)$ is differentiable almost everywhere.

As is well-known, the incentive compatibility is characterized as follows.

Lemma 1. *A direct revelation mechanism $\langle x(\cdot), r(\cdot)\rangle$ is incentive compatible if and only if $x(\cdot)$ is non-decreasing and $\dot{r}(\theta) = u_\theta(x(\theta), \theta) - \dot{\bar{\pi}}(\theta)$ for every $\theta \in \Theta$.*

The purpose of the paper is to show how to construct a price schedule satisfying the voluntary implementability for any feasible direct revelation mechanisms. The following theorem states that the reverse of Theorem 1 actually holds.

Theorem 2 (Taxation Principle)**.** *If a direct revelation mechanism $\langle x(\cdot), r(\cdot)\rangle$ is incentive compatible and individually rational, then it is voluntarily implementable.*

Proof. For each $x \in X$, define

$$t(x) = \max\left[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x, \hat{\theta}) \mid \hat{\theta} \in \Theta\right].$$

Step 1. $x(\theta) \in \operatorname{argmax}[u(x,\theta) - t(x) \mid x \in X]$ *for every* $\theta \in \Theta$.

Proof of Step 1. It suffices to show that $\pi_t(\theta) = u(x(\theta), \theta) - t(x(\theta))$ for every $\theta \in \Theta$. Consider any $\theta \in \Theta$.

Claim 1. $u(x(\theta),\theta) - t(x(\theta)) = r(\theta) + \bar{\pi}(\theta)$.

Proof of Claim 1. Let $h(x,\hat{\theta}) = u(x,\hat{\theta}) - (r(\hat{\theta}) + \bar{\pi}(\hat{\theta}))$. Using the envelope condition $\dot{r}(\theta) = u_\theta(x(\theta),\theta) - \dot{\bar{\pi}}(\theta)$ in Lemma 1, I obtain $h_\theta(x,\hat{\theta}) = u_\theta(x,\hat{\theta}) - u_\theta(x(\hat{\theta}),\hat{\theta}) = \int_{x(\hat{\theta})}^{x} u_{x\theta}(z,\hat{\theta})dz$. The first-order condition $0 = h_\theta(x(\theta),\hat{\theta})$ yields $x(\theta) = x(\hat{\theta})$ by the single-crossing property. The second-order condition at $x = x(\theta)$ is written as $h_{\theta\theta}(x(\theta),\hat{\theta}) = u_{\theta\theta}(x(\theta),\hat{\theta}) - u_{x\theta}(x(\hat{\theta}),\hat{\theta}) \cdot \dot{x}(\hat{\theta}) - u_{\theta\theta}(x(\hat{\theta}),\hat{\theta}) = -u_{x\theta}(x(\hat{\theta}),\hat{\theta}) \cdot \dot{x}(\hat{\theta}) \leqslant 0$ almost everywhere. Therefore, $h(x(\theta),\hat{\theta})$ is maximized at $\hat{\theta} = \theta$. Thus, $t(x(\theta)) = h(x(\theta),\theta) = u(x(\theta),\theta) - (r(\theta) + \bar{\pi}(\theta))$. This establishes the claim.

Claim 2. $\pi_t(\theta) = r(\theta) + \bar{\pi}(\theta)$.

Proof of Claim 2. Consider any $x \in X$. By the definition of $t(x)$, I see that $t(x) \geqslant -(r(\theta) + \bar{\pi}(\theta)) + u(x, \theta)$, which implies that $u(x, \theta) - t(x) \leqslant r(\theta) + \bar{\pi}(\theta)$. Since x was arbitrary, it follows that $\pi_t(\theta) \leqslant r(\theta) + \bar{\pi}(\theta)$. It remains to show that $\pi_t(\theta) \geqslant r(\theta) + \bar{\pi}(\theta)$. I see that $\pi_t(\theta) - (r(\theta) + \bar{\pi}(\theta)) = \max[u(x, \theta) - t(x) \mid x \in X] - (r(\theta) + \bar{\pi}(\theta)) \geqslant u(x(\theta), \theta) - t(x(\theta)) - (r(\theta) + \bar{\pi}(\theta)) = 0$, where the last equality follows from Claim 1. Therefore, $\pi_t(\theta) \geqslant r(\theta) + \bar{\pi}(\theta)$. This establishes the claim.

By Claims 1 and 2, $\pi_t(\theta) = u(x(\theta), \theta) - t(x(\theta))$. This establishes the step.

Step 2. $r(\theta) = \pi_t(\theta) - \bar{\pi}(\theta) \geqslant 0$.

Proof of Step 2. The equality is immediate from Claim 2. By the individual rationality, $r(\theta) \geqslant 0$ for every $\theta \in \Theta$. By Step 1, $\pi_t(\theta) = u(x(\theta), \theta) - t(x(\theta)) = r(\theta) + \bar{\pi}(\theta) \geqslant \bar{\pi}(\theta)$. This establishes the inequality and the step.

Steps 1 and 2 establish the theorem. □

I have established the following characterization result.

Theorem 3. *A direct revelation mechanism $\langle x(\cdot), r(\cdot)\rangle$ is incentive compatible and individually rational if and only if it is voluntarily implementable.*

Proof. Immediate from Theorems 1 and 2. □

I have constructed a particular price schedule for voluntary implementation in the proof of Theorem 2. The following remark states that, without loss of generality, I can restrict attention to the price schedule $t(x) = \max\left[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x, \hat{\theta}) \mid \hat{\theta} \in \Theta\right]$, and is hereafter referred to as the *optimal* price schedule.

Proposition 1. *If a price schedule voluntarily implements a direct revelation mechanism $\langle x(\cdot), r(\cdot)\rangle$, then such a price schedule is uniquely determined over the range of the decision rule.*

Proof. Suppose that $t(\cdot)$ and $\tilde{t}(\cdot)$ voluntarily implements $x(\cdot)$. Then, for every $\theta \in \Theta$, $u(x(\theta), \theta) - t(x(\theta)) - \bar{\pi}(\theta) = r(\theta) = u(x(\theta), \theta) - \tilde{t}(x(\theta)) - \bar{\pi}(\theta)$, which yields that $t(x(\theta)) = \tilde{t}(x(\theta))$. Therefore, $t(x) = \tilde{t}(x)$ for every $x \in [x(\underline{\theta}), x(\overline{\theta})]$. □

By the previous proposition, whenever I mention the *optimal* price schedule, it is of the form: for each $x \in [x(\underline{\theta}), x(\overline{\theta})]$,

$$t(x) = -(r(\psi(x)) + \bar{\pi}(\psi(x))) + u(x, \psi(x)),$$

$$\text{where } \psi(x) \in \operatorname{argmax}[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x, \hat{\theta}) \mid \hat{\theta} \in \Theta].$$

To end this section, I want to summarize some properties of the optimal price schedule constructed in the proof of Theorem 2.

Proposition 2. *The optimal price schedule $t(\cdot)$ is continuous and increasing.*

Proof. The optimal price schedule $t(x) = \max[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x,\hat{\theta}) \mid \hat{\theta} \in \Theta]$ is continuous by the maximum theorem because the type space is compact. It remains to show the monotonicity of $t(\cdot)$. Let $\psi(x) \in \text{argmax}[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x,\hat{\theta}) \mid \hat{\theta} \in \Theta]$. If $x > y$, then

$$\begin{aligned} t(x) - t(y) &\geqslant -(r(\psi(y)) + \bar{\pi}(\psi(y))) + u(x,\psi(y)) \\ &\quad -[-(r(\psi(y)) + \bar{\pi}(\psi(y))) + u(y,\psi(y))] \\ &= u(x,\psi(y)) - u(y,\psi(y)) = \int_y^x u_x(z,\psi(y))dz > 0 \end{aligned}$$

because $u_x(x,\theta) > 0$. Hence, $t(x) > t(y)$. This establishes the proposition. □

Since the price schedule $t(x) = \max[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x,\hat{\theta}) \mid \hat{\theta} \in \Theta]$ is increasing by Proposition 2, it is differentiable almost everywhere, and it has kinks at non-differentiable points.

Theorem 4. *The optimal price schedule $t(\cdot)$ satisfies the following envelope condition:*

$$t'(x) = u_x(x,\psi(x)),$$

$$\textit{where } \psi(x) \in \textit{argmax}[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x,\hat{\theta}) \mid \hat{\theta} \in \Theta].$$

Proof. Immediate from applying the envelope theorem to $t(x) = \max[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x,\hat{\theta}) \mid \hat{\theta} \in \Theta]$. □

The standard approach to the screening problems (when the reservation utility is not type-dependent) employs a non-decreasing decision rule $x : \Theta \to X$. Nöldeke and Samuelson (2007) reformulate principal-agent problems as choosing a non-decreasing function $\psi : X \to \Theta$ satisfying the envelope condition $t'(x) = u_x(x,\psi(x))$ shown in the previous proposition.

In addition, Maskin and Riley (1984, Proposition 3) also derive the optimal price schedule in an extended framework of Mussa and Rosen (1978). Their result can be obtained as a corollary of Theorem 2.

4 Duality in Nonliear Pricies

In this section, I shall derive further properties of the optimal price schedule $t(\cdot)$. In Theorem 2, the optimal price schedule is obtained as $t(x) = -(r(\psi(x)) + \bar{\pi}(\psi(x))) + u(x,\psi(x))$, where $\psi(x) \in \text{argmax}[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x,\hat{\theta}) \mid \hat{\theta} \in \Theta]$. Let us call $\Gamma(\cdot) = \text{argmax}[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(\cdot,\hat{\theta}) \mid \hat{\theta} \in \Theta]$ the *price adjustment correspondence*. It will be shown that a selection $\psi : X \to \Theta$ from $\Gamma : X \twoheadrightarrow \Theta$ can be referred to as a *type assignment function*.

4.1 Price Adjustment Correspondence

Any non-decreasing decision rule $x(\cdot)$ is implementable by the optimal price schedule. Intuitively speaking, the optimal price schedule calculates the optimal marginal price or the optimal targeted willingness to pay for each $x \in X$.

Definition 4. A decision rule $x(\cdot)$ *has a bunch at* $y \in X$ if $x(\theta) = y$ over some $[\theta_1, \theta_2] \subseteq \Theta$ with $\theta_1 < \theta_2$.

Lemma 2. *Let $\langle x(\cdot), r(\cdot) \rangle$ be an incentive compatible and individually rational direct revelation mechanism. Then,*
(1) $\Gamma(y)$ is a compact subset of Θ for every $y \in X$,
(2) the composite $\Gamma \circ x : \Theta \twoheadrightarrow \Theta$ is self-belonging, that is, $\theta \in \Gamma(x(\theta))$ for every $\theta \in \Theta$,
(3) $\Gamma(y) = \text{argmax}\left[u(y,\hat{\theta}) - \int_{\theta^*}^{\hat{\theta}} u_\theta(x(s),s)ds \mid \hat{\theta} \in \Theta\right] - \bar{\pi}(\theta^*)$ where $\theta^* \in \Theta$ such that $r(\theta^*) = 0$ for every $y \in X$.

Proof. (1) Immediate from Berge's maximum theorem.
(3) Since $\dot{r}(\theta) = u_\theta(x(\theta),\theta) - \dot{\bar{\pi}}(\theta)$, it follows that $r(\theta) = \int_{\theta^*}^{\theta} [u_\theta(x(s),s) - \dot{\bar{\pi}}(s)]ds$, where $\theta^* \in \Theta$ such that $r(\theta^*) = 0$. Recall that $\Gamma(x) = \text{argmax}\left[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x,\hat{\theta}) \mid \hat{\theta} \in \Theta\right]$. The expression inside square brackets can be written as

$$\begin{aligned}
-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x,\hat{\theta}) &= -\int_{\theta^*}^{\hat{\theta}} [u_\theta(x(s),s) - \dot{\bar{\pi}}(s)]ds - \bar{\pi}(\hat{\theta}) + u(x,\hat{\theta}) \\
&= -\int_{\theta^*}^{\hat{\theta}} u_\theta(x(s),s)ds + [\bar{\pi}(\hat{\theta}) - \bar{\pi}(\theta^*)] - \bar{\pi}(\hat{\theta}) + u(x,\hat{\theta}) \\
&= u(x,\hat{\theta}) - \int_{\theta^*}^{\hat{\theta}} u_\theta(x(s),s)ds - \bar{\pi}(\theta^*),
\end{aligned}$$

and hence,

$$\Gamma(x) = \text{argmax}\left[u(x,\hat{\theta}) - \int_{\theta^*}^{\hat{\theta}} u_\theta(x(s),s)ds \mid \hat{\theta} \in \Theta\right] - \bar{\pi}(\theta^*).$$

(2) Suppose, by way of contradiction, that $\theta \notin \Gamma(x(\theta))$. By definition, $\psi(x(\theta)) \in \Gamma(x(\theta))$. Since $\theta \in \Theta$, it must be the case that

$$u(x(\theta), \psi(x(\theta))) - \int_{\theta^*}^{\psi(x(\theta))} u_\theta(x(s),s)ds > u(x(\theta),\theta) - \int_{\theta^*}^{\theta} u_\theta(x(s),s)ds.$$

There are two possible cases to be considered. If $\theta \geqslant \psi(x(\theta))$, then this inequality gives

$$\begin{aligned}
0 &> \int_{\psi(x(\theta))}^{\theta} u_\theta(x(\theta),s)ds - \int_{\psi(x(\theta))}^{\theta} u_\theta(x(s),s)ds \\
&\geqslant \int_{\psi(x(\theta))}^{\theta} u_\theta(x(\theta),s)ds - \int_{\psi(x(\theta))}^{\theta} u_\theta(x(\theta),s)ds = 0.
\end{aligned}$$

This is a contradiction. If $\theta \leqslant \psi(x(\theta))$, then the above inequality gives

$$
\begin{aligned}
0 &> \int_{\psi(x(\theta))}^{\theta} u_\theta(x(\theta),s)ds + \int_{\theta}^{\psi(x(\theta))} u_\theta(x(s),s)ds \\
&\geqslant -\int_{\theta}^{\psi(x(\theta))} u_\theta(x(\theta),s)ds + \int_{\theta}^{\psi(x(\theta))} u_\theta(x(\theta),s)ds = 0.
\end{aligned}
$$

This is a contradiction. Therefore, it must be the case $\theta \in \Gamma(x(\theta))$. □

4.2 Type-Assignment Function in Relation to Goldman et al. (1984)

The following proposition states that the construction of the optimal price schedule actually involves the inverse of decision rules.

Lemma 3. *Let $\langle x(\cdot), r(\cdot)\rangle$ be an incentive compatible and individually rational direct revelation mechanism. For every selection $\psi(\cdot) \in \Gamma(\cdot)$,*
(1) the composite $x \circ \psi : X \to X$ is the identity,
(2) the composite $\psi \circ x : \Theta \to \Theta$ is the identity at which $x(\cdot)$ has no bunch.

Proof. (1) Let $y \in X$. Any interior optimum $\psi(y)$ yields the following first-order condition:

$$
\begin{aligned}
0 &= -(\dot{r}(\psi(y)) + \dot{\bar{\pi}}(\psi(y))) + u_\theta(y, \psi(y)) \\
&= -u_\theta(x(\psi(y)), \psi(y)) + u_\theta(y, \psi(y)) \\
&= \int_{x(\psi(y))}^{y} u_{x\theta}(z, \psi(y))dz,
\end{aligned}
$$

and hence $y = x(\psi(y)) = (x \circ \psi)(y)$. This establishes the assertion.
(2) Without loss of generality, I may choose $\psi(x) = \min \Gamma(x)$. Since $\theta \in \Gamma(x(\theta))$ and $\psi(x(\theta)) \in \Gamma(x(\theta))$, it follows that $\psi(x(\theta)) \leqslant \theta$. It remains to show that this holds with equality. Suppose, by way of contradiction, that $\theta > \psi(x(\theta))$. If $x(\cdot)$ has no bunch, then it is strictly increasing. This yields that $x(\theta) > x(\psi(x(\theta))) = (x \circ \psi)(x(\theta)) = x(\theta)$, a contradiction. Therefore, it must be the case $\psi(x(\theta)) = \theta$. This establishes the assertion. □

The above observation holds regardless of the differentiability of $t(\cdot)$ at y.[4]

Theorem 5. *Let $\langle x(\cdot), r(\cdot)\rangle$ be an incentive compatible and individually rational direct revelation mechanism. Then, $\Gamma(y) = \{\theta \in \Theta \mid x(\theta) = y\}$ for every $y \in X$.*

[4] The second assertion can be proved, provided that the optimal price schedule $t(\cdot)$ is differentiable. Recall that the implementability constraint is satisfied under the price schedule $t(x) = \max\,[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x, \hat{\theta}) \mid \hat{\theta} \in \Theta]$, that is, $x(\theta) \in \operatorname{argmax}[u(x,\theta) - t(x) \mid x \in X]$. The first-order condition is given by $0 = u_x(x(\theta), \theta) - t'(x(\theta))$. By Theorem 4, $t'(x) = u_x(x, \psi(x))$. Thus, $u_x(x(\theta), \theta) = u_x(x(\theta), \psi(x(\theta)))$. By the strict single-crossing property $u_{x\theta}(x, \theta) > 0$, it must be the case $\theta = \psi(x(\theta)) = (\psi \circ x)(\theta)$.

Proof. It suffices to consider $y \in X$ at which $x(\cdot)$ has a bunch. Since $x(\cdot)$ is non-decreasing by Proposition 1, it must be the case that $\{\theta \in \Theta \mid x(\theta) = y\} = [\theta_1, \theta_2]$ for some $\theta_1, \theta_2 \in \Theta$ with $\theta_1 < \theta_2$. Since $x(\theta) = y$ for every $\theta \in [\theta_1, \theta_2]$, it follows from Lemma 2(2) that $\theta \in \Gamma(x(\theta)) = \Gamma(y)$, which implies that $[\theta_1, \theta_2] \subseteq \Gamma(y)$. It remains to show that $\Gamma(y) \subseteq [\theta_1, \theta_2]$. Let $\theta \in \Gamma(y)$. Then, $\theta = \psi(y)$ for some selection $\psi(y)$ from $\Gamma(y)$. Then, $x(\theta) = x(\psi(y)) = (x \circ \psi)(y) = y$, where the last equality follows from Lemma 3(1). This implies that $\theta \in [\theta_1, \theta_2]$. Therefore, $\Gamma(y) \subseteq [\theta_1, \theta_2]$. This establishes the proposition. □

Corollary 1. *Let $\langle x(\cdot), r(\cdot)\rangle$ be an incentive compatible and individually rational direct revelation mechanism. Then, $\Gamma(y) = \{x^{-1}(y)\}$ for every $y \in X$ at which $x(\cdot)$ has no bunch.*

Instead of using such a non-decreasing decision rule $x(\cdot)$, Goldman et al. (1984) consider a dual problem as choosing a non-decreasing type assignment function $\psi : X \to \Theta$ for a Ramsey pricing problem. Corollary 1 shows that the type-assignment function used in Goldman et al. (1984) is a selection from the compact-valued correspondence $\Gamma(\cdot)$ indeed.

Example 1 (Bunching)**.** Just for an illustration, I shall derive the price function for some decision rule in which the reservation utility is zero for every type. Let $\overline{\theta} > 5a > 0$ and $b > 0$. Let θ be distributed on $[\underline{\theta}, \overline{\theta}]$, where $\underline{\theta} = \frac{1}{2}(\overline{\theta} + a)$. Let $\theta_a = \frac{1}{4}(3\overline{\theta} - \underline{\theta})$ and $\theta_b = \frac{1}{4}(5\overline{\theta} - 3\underline{\theta})$. Substituting $\underline{\theta} = \frac{1}{2}(\overline{\theta} + a)$, I see that $\theta_a = \frac{1}{8}(5\overline{\theta} - a)$ and $\theta_b = \frac{1}{8}(7\overline{\theta} - 3a)$. Then, $\theta_a < \theta_b$ if and only if $\overline{\theta} > a$. Consider the following decision rule depicted in Figure 2:

$$x(\theta) = \begin{cases} \frac{1}{b}(2\theta - (\overline{\theta} + a)) & \text{for } \theta \in [\underline{\theta}, \theta_a), \\ \frac{1}{2b}(\overline{\theta} - 2a - \underline{\theta}) & \text{for } \theta \in [\theta_a, \theta_b], \\ \frac{1}{2b}(4\theta - (3\overline{\theta} + a)) & \text{for } \theta \in (\theta_b, \overline{\theta}]. \end{cases}$$

There is pooling at $y = \frac{1}{2b}(\overline{\theta} - 2a - \underline{\theta})$.

According to Corollary 1, the price schedule implementing the decision rule is constructed through the inverse of the decision rule except the bunching point. For each $x < \frac{1}{2b}(\overline{\theta} - 2a - \underline{\theta})$, the type-assignment $\psi(x) = \frac{1}{2}(bx + \overline{\theta} + a)$ can be found from the equation $x = \frac{1}{2b}(2\psi(x) - (\overline{\theta} + a))$. Similarly, for each $x > \frac{1}{2b}(\overline{\theta} - 2a - \underline{\theta})$, the type-assignment $\psi(x) = \frac{1}{2}(2bx + 3\overline{\theta} + a)$ can be found from the equation $x = \frac{1}{2b}(4\psi(x) - (3\overline{\theta} + a))$. Figure 3 illustrates the price schedule $t(\cdot)$ that implements $x(\cdot)$. The corresponding marginal price function is given by

$$t'(x) = \begin{cases} \frac{1}{2}(bx + \overline{\theta} + a) & \text{for } x < \frac{1}{2b}(\overline{\theta} - 2a - \underline{\theta}), \\ \frac{1}{4}(2bx + 3\overline{\theta} + a) & \text{for } x > \frac{1}{2b}(\overline{\theta} - 2a - \underline{\theta}). \end{cases}$$

There is a jump in the optimal marginal price function at $y = \frac{1}{2b}(\overline{\theta} - 2a - \underline{\theta})$: $\lim_{x \uparrow y} t'(x) = \theta_a < \theta_b = \lim_{x \downarrow y} t'(x)$. Moreover, $\lim_{x \downarrow 0} t'(x) = \underline{\theta}$. The price schedule $t : X \to \mathbb{R}$ is anonymous. It is defined over the product line X, rather than over the type space Θ.

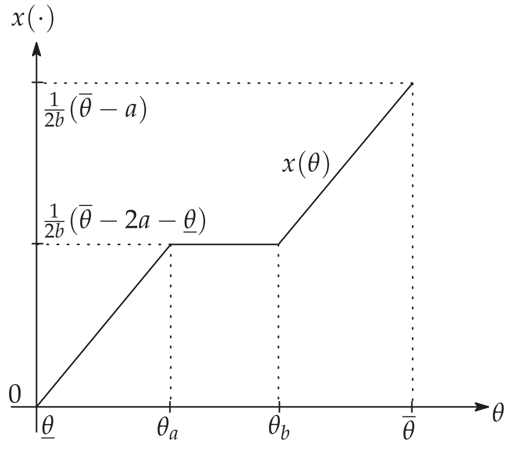

Figure 2: Pooling over $[\theta_a, \theta_b]$

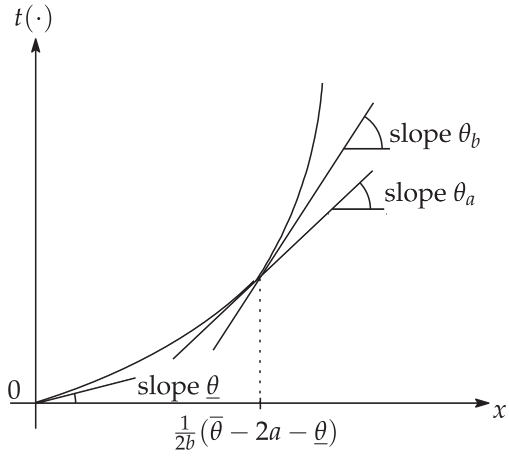

Figure 3: Kink at $\frac{1}{2b}(\overline{\theta} - 2a - \underline{\theta})$

5 Applications

5.1 The Optimality of Block Tariffs

Block Tariffs are simple, but it is trivial that the principal could increase his expected profit beyond that achievable with an optimal block tariff by choosing a more complex nonlinear pricing scheme. A *two-part tariff* $t(x) = px + q$ can be considered as a special case of a block tariff. The current model corresponds to second-degree price discrimination due to Pigou (1932). In the context of first-degree price discrimination, the use of two-part tariffs has been discussed. However, the optimality of a two-part tariff or a block tariff in general has not been explored in the context of second-degree price discrimination because of the intensive use of direct revelation mechanisms.

Definition 5. A *block tariff* is a piecewise linear price schedule.

The following theorem provides a necessary and sufficient condition for that the optimal price schedule obtained in Theorem 2 takes the form of a block tariff.

Theorem 6 (Block Tariff)**.** *Let $\langle x(\cdot), r(\cdot)\rangle$ be an incentive compatible and individually rational direct revelation mechanism. Then, the following statements are equivalent:*
(1) the optimal price schedule $t(\cdot)$ takes the form of a block tariff.
(2) $\frac{d}{d\theta}u_x(x(\theta), \theta) = 0$ except at $\theta \in \Theta$ where $x(\cdot)$ has no bunch at $x(\theta)$.

Proof. Recall the envelope condition $t'(x) = u_x(x, \psi(x))$, where $\psi(x) \in \text{argmax}[-(r(\hat{\theta}) + \bar{\pi}(\hat{\theta})) + u(x, \hat{\theta}) \mid \hat{\theta} \in \Theta]$. In addition, $\psi(y) = x^{-1}(y)$ if and only if $x(\cdot)$ has no bunch at $y \in X$ by Lemma 3. Differentiating $t'(\cdot)$ to obtain

$$
\begin{aligned}
t''(y) &= u_{xx}(y,\psi(y)) + u_{x\theta}(y,\psi(y)) \cdot \psi'(y) \\
&= u_{xx}(x(\theta),\theta) + u_{x\theta}(x(\theta),\theta) \cdot \tfrac{1}{\dot{x}(\theta)} \\
&= \tfrac{1}{\dot{x}(\theta)} \left(u_{xx}(x(\theta),\theta) \cdot \dot{x}(\theta) + u_{x\theta}(x(\theta),\theta) \right) \\
&= \tfrac{1}{\dot{x}(\theta)} \cdot \tfrac{d}{d\theta} u_x(x(\theta),\theta).
\end{aligned}
$$

This implies that for every $x \in X$ where the price schedule is twice differentiable, it must be the case that $t''(x) = 0$ is equivalent to saying that $\frac{d}{d\theta} u_x(x(\theta),\theta) = 0$, using the duality between $\theta = \psi(x)$ and $x = x(\theta)$. This establishes the theorem. □

As a conclusion of the previous theorem, the emergence of a block tariff as a solution to principal-agent problems depends on the form of the utility function of the agent. In this section, I consider a commonly used utility function of the agent to check whether a block tariff can realize a given decision rule.

Proposition 3. *Let $\langle x(\cdot), r(\cdot) \rangle$ be an incentive compatible and individually rational direct revelation mechanism. If the utiliy function of the agent takes of the form $u(x,\theta) = \theta v(x)$, where $v : X \to \mathbb{R}$ is strictly increasing and concave, then the optimal price schedule $t(\cdot)$ takes the form of a block tariff if and only if $\frac{d}{dy}(\psi(y)v'(y)) = 0$, where $\psi(y) = x^{-1}(y)$ except at bunching points of $x(\cdot)$.*

Proof. The expression of $t''(x)$ obtained in the proof of Theorem 6 can be written as

$$
\begin{aligned}
t''(y) &= u_{xx}(y,\psi(y)) + u_{x\theta}(y,\psi(y)) \cdot \psi'(y) \\
&= \psi(y) \cdot v''(y) + v'(y) \cdot \psi'(y) \\
&= \tfrac{d}{dy}(\psi(y)v'(y))
\end{aligned}
$$

for every $y \in X$ except at bunching points of $x(\cdot)$. The equivalence result is immediate from Theorem 6. □

5.2 Quality Premia or Quantity Discounts

Recall that the following expression for the second-derivative of the optimal price schedule shown in the proof of Theorem 6:

$$
\text{sign } t''(y) = \text{sign } [u_{xx}(x(\theta),\theta) \cdot \dot{x}(\theta) + u_{x\theta}(x(\theta),\theta)], \text{ whenever } x(\theta) \text{ has no bunch at } y.
$$

This yields that the sign of $t''(y)$ is determined by the relationship between $\mid u_{xx}(y,\theta) \cdot \dot{x}(\theta) \mid$ and $\mid u_{x\theta}(y,\theta) \mid$, where $x(\theta) = y$. A few findings based on this observation can be stated as follows.

Proposition 4. *Let $\langle x(\cdot), r(\cdot)\rangle$ be an incentive compatible and individually rational direct revelation mechanism. Then,*
(1) if the optimal price schedule $t(\cdot)$ is of the form of $t(x) = px + q$, then it must be the case that $u_{xx}(x, \theta) < 0$ for every x and θ,
(2) if the optimal price schedule $t(\cdot)$ is strictly concave, then it must be the case that $u_{xx}(x, \theta) < 0$ for every x and θ,
(3) if $u_{xx}(x, \theta) = 0$ for every x and θ, then the optimal price schedule $t(\cdot)$ must be strictly convex.

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